CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

EXAMINATION FOR THE AWARD OF DEGREE OF MASTER OF SCIENCE IN MATHEMATICS (PURE)

MATH 814: OPERATOR THEORY 1

STREAMS: **TIME: 3 HOURS**

DAY/DATE: WEDNESDAY 13/12/2017 **2.30 P.M – 5.30 P.M**

INSTRUCTIONS:

- **Answer any three questions**
- **Do not write on the question paper**

QUESTION ONE: (20 MARKS)

(a) (i) Prove that if S and T are two positive self adjoint linear operators on complex Hilbert space H then their sum is also positive. (1 mark)

(ii) Hence prove that if two bounded self adjoint linear operators S and T on complex Hilbert space H are positive and commutate, then their product is positive. (8marks)

(b) Define a positive square root of a self adjoint linear operator P Hilbert space H. Hence show that for a self adjoint bounded operator P,

$$\| Px \| \leq \| P \|^{\frac{1}{2}} \langle Px, x \rangle^{\frac{1}{2}}$$ (3marks)

(c) Let $P \in B(X)$. Show that PP^* and P^*P are positive self adjoints and their spectra are real and does not contain negative values. (3 marks)

(d) If $T_n, S_n \in B(X)\ \forall\ n \in \mathbb{N}\ and\ T, S \in B(X)$ such that $T_n \to T, S_n \to S$. Prove that $T_nS_n \to TS$ (5 marks)

QUESTION TWO: (20 MARKS)

(a) Let Pbe a projection on a Hilbert space H. Prove that

(i) $\| P \| \leq 1$: $\| P \| = 1 \; iff \; P(H) \neq \{0\}$ (3 marks)

(ii) There exists a closed linear subspace M of H such that $P = P_M$ or $P_M(H) = M$

(3 marks)

(b) (i) Let $P_1 \; and \; P_1$ be projections on a Hilbert space H. Then prove that their sum $P = P_1 + P_2$ is a projection on H iff $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$ are orthogonal. (4 marks)

(ii) Prove that a bounded linear operator $P: X \to X$ on a Hilbert space H is a projection iff P is self adjoint and idempotent. (6 marks)

(c) Let H be a Hilbert space, M a linear closed subspace of H and $y \in H \backslash M$. Prove that there exists a unique projection $P_y \in M$ such that $\| y - P_y \| = Inf\{\| y - x \| : x \in M\}$ (4 marks)

QUESTION THREE: (20 MARKS)

(a) Let U be a partial Isometry in $B(H)$. Show that U^*U is an orthogonal projection. (4 marks)

(b) Let H be a Hilbert space. Prove that the following statements on a Unitary linear operator U are equivalent

(i) $U = UU^*U$

(ii) $P = U^*U$ is a projection

(iii) $U/ker^{\perp}U$ is an isometry (5 marks)

(c) (i) Suppose $x_n(k = 1,2, \dots, n)$ and $y_j(j = 1,2, \dots, m)$ be elements in an inner product space $(\mathcal{Y}, <,>)$and $\alpha_k, \beta_j \in K$. Show that

$\langle \sum_{k=1}^{n} \alpha_k x_k \, , \, \sum_{j=1}^{m} \beta_j y_j \rangle = \sum_{k=1}^{n} \sum_{j=1}^{m} \alpha_k \bar{\beta}_j \langle x_k, y_j \rangle$ (4 marks)

(ii) Prove that every inner product space is a normed linear space with respect to the norm determined by the inner product function (4 marks)

(d) State and prove the Cauchy-Bunyakowski-Schwarz inequality in inner product spaces

(4 marks)

QUESTION FOUR: (20 MARKS)

(a) Let $A = \begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix}$. Determine the spectrum and eigen space of A. (4 marks)

(b) Prove that all matrices representing a given linear operator $T: X \to X$ on a finite dimensional normed space X relative to various bases for X have the same eigen values. (7 marks)

(c) Define the numerical range$Num(T)$ of an operator T on a Hilbert space H. Hence prove that for any $T \in B(X)$the spectrum of T is contained in the closure of the numerical range

(4 marks)

(d) (i) When is a bounded linear operator $T: X \to X$ on a normed space X said to satisfy the Fredholm alternative? (4 marks)

(ii) Define a Hilbert- Schmidt norm of an operator $T \in B(H_1, H_2)$ where H_1, H_2 are separable Hilbert spaces. (1 mark)
