

CHUKA

UNIVERSITY



UNIVERSITY EXAMINATIONS

**THIRD YEAR EXAMINATION FOR THE DEGREE OF BACHELORS OF EDUCATION
SCIENCE/ARTS, BACHELORS OF SCIENCE GENERAL, BACHELORS OF SCIENCE IN
MATHEMATICS, BACHELORS OF ARTS(MATHS-ECONS)**
(JAN-APRIL 2023)
MATH 303: REAL ANALYSIS II

STREAMS: ````as above```` Y3S2

TIME: 2HRS

DAY/DATE:

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INSTRUCTIONS:

- Answer question **ONE** and **TWO** other questions
- Do not write on the question paper
- Write your answers legibly and use your time wisely

QUESTION ONE: (30 MARKS)

- (a) Discuss the following concepts as used in analysis
- A portion of a closed interval $[a, b]$ (1 mark)
 - The Riemanns' upper sum and lower sum of the function f (3 marks)
 - The Riemann Stieltjes integrable function on $[a, b]$ (3 marks)
- (b) Prove that if $f_n \neq 0 \forall n \in N$ is a sequence of functions which converges to the limit $f \neq 0$ then $\frac{1}{f_n}$ is convergent to $\frac{1}{f}$ (as $n \rightarrow \infty$) (5 marks)
- (c) Define a Step function. Hence show that a step function is always Riemann Integrable. (4 marks)
- (d) By sketching the graphs of the function $f(x) = \log_5 x$ and $f(x) = \log_{\frac{1}{5}} x$ on the same axis, state the relationship between the two graphs (5 marks)

- (e) Let $f(x) = x$ for $a \leq x \leq b$ and define α on $[a, b]$ by $\alpha x = 0$ for $a \leq x \leq b$ with $\alpha(b) = c$. If (P, t) is a tagged partition of $[a, b]$ with $P = \{x_0, x_1, x_2, \dots, x_n\}$, define $S(P, t, f, \alpha) = t_n c$. Show that $\int_a^b x d\alpha(x) = bc$ (3 marks)
- (f) Distinguish between pointwise convergence and uniform convergence of sequences of functions (2 marks)
- (g) Distinguish an odd function and an even function. Give an example in each case. Hence prove that if the function $f(x)$ is even and $g(x)$ is odd, then the product $f(x)g(x)$ is odd (4 marks)

QUESTION TWO: (20 MARKS)

- (a) State and prove the Comparison test (Weierstrauss M-test) for convergence of real valued infinite series of functions. (9 marks)
- (b) Hence by using Comparison theorem, prove that for a series of positive terms $\sum_{n \in \mathbb{N}} f_n$ such that $\lim_{n \rightarrow \infty} \left(\frac{f_{n+1}}{f_n} \right) = l$, if $l < 1$ then series is convergent (6 marks)
- (a) (i) State the Cauchy criterion for convergence of a series of functions (2 marks)
- (ii) Hence prove that for a series of functions $\sum_{n \in \mathbb{N}} f_n$ on \mathbf{K} , this series only converges if $\forall \varepsilon > 0 \exists N(\varepsilon) \in \mathbb{N}: |\sum_{k=m}^n f_k| < \varepsilon$ for every $n \geq m \geq N(\varepsilon)$ (3 marks)

QUESTION THREE: (20 MARKS)

- (a) Describe in details how each of the following integrals are used to determine the area of the function f on the interval $[a, b]$
- (i) Riemann Integral (4 marks)
- (ii) Riemann-Stieltjes Integral (3 marks)
- (c) Illustrate whether a Dirichlet function on the interval $[a, b]$ is Riemann Integrable or not. (4 marks)
- (d) Show that the function $f(x) = 5x$ is Riemann integrable on $[0, 1]$ and that $\int_0^1 f(x) = 2.5$ (9 marks)

QUESTION FOUR: (20 MARKS)

(a) Define the Fourier series of the function $f(x)$ on the interval $-l$ to l . (3 marks)

(b) Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} 1 & -\pi < x < 0 \\ 0 & 0 < x < \pi \end{cases}$$

(i) Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$ (2 marks)

(ii) Show that the Fourier series for $f(x)$ in the interval $-\pi < x < \pi$ is

$$\frac{1}{2} - \frac{2}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right]$$

(8 marks)

(c) State and prove the Cauchy's Root Test for convergence of functions of an infinite series. (7 marks)

QUESTION FIVE: (20 MARKS)

(a) Define an absolutely convergent series. Hence prove that an absolute convergent series of functions in (K, d) is necessarily convergent, however using an appropriate example illustrate that the converse of this is not true. (8 marks)

(b) State and prove the Intermediate Mean Value Theorem (6 marks)

(c) Let $f: [p, q] \rightarrow R$ be continuous and that $f(p)f(q) < 0$, then there exists $c \in (p, q)$ such that $f(c) = 0$. (6 marks)