

CHUKA



UNIVERSITY

## UNIVERSITY EXAMINATIONS

**EXAMINATION FOR THE AWARD OF DEGREE OF BACHELORS OF  
EDUCATION (SCIENCE /ARTS), BACHELORS OF SCIENCE GENERAL,  
BACHELORS OF ARTS(MATHS-ECONS), BACHELORS OF  
SCIENCE(MATHEMATICS)**

MATH 301: LINEAR ALGEBRA II

STREAMS: As above

TIME: 2 HOURS

DAY/DATE: TUESDAY 19/12/2023

8.30 A.M. – 10.30 A.M.

**INSTRUCTIONS:**

- Answer Question **ONE** and any other **TWO** Questions

**QUESTION ONE: (30 MARKS)**

- a) Find the matrix P that represents the usual inner product on  $C^3$  relative to the basis  $\{1, i, 1 - i\}$   
(5 marks)

- b) Given that  $A = \begin{bmatrix} 2 & -3 & 3 \\ 3 & -4 & 3 \\ 6 & -6 & 5 \end{bmatrix}$ , find the eigenvalues of  $A^3$  (5 marks)

- c) Find the symmetric matrix that correspond to the following quadratic form  
 $q(x, y, z) = xy + y^2 - 4xz + z^2$  (3 marks)
- d) Let A be an nxn matrix over a field K. show that the mapping  $f(X, Y) = X^T AY$  is a bilinear form on  $K^n$  (3 marks)
- e) Prove that similar matrices have the same characteristic polynomial. (3 marks)

f) State how elementary row operations affect the determinant of a square matrix, hence or otherwise show that if two rows are the same the determinant is zero. (4 marks)

g) Show that if  $A = \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix}$ , then find  $A^{10}$  (3 marks)

h) Find the minimal polynomial of the matrix  $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$  (4 marks)

**QUESTION TWO (20 MARKS)**

a) Let  $f$  be a bilinear form on  $R^2$  defined by

$$f[(x_1, x_2), (y_1, y_2)] = 3x_1y_1 - 2x_1y_2 + 4x_2y_1 - x_2y_2. \text{ Find}$$

- i. The matrix A of  $f$  in the basis  $\{u_1 = (1,0), u_2 = (1,1)\}$
- ii. The matrix B of  $f$  in the basis  $\{v_1 = (2,1), v_2 = (1,-1)\}$
- iii. The change of basis matrix P from the basis  $\{u_i\}$  to the basis  $\{v_i\}$  and verify that  $B = P^T A P$ .

(12 marks)

b) Let A be the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 5 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

Apply diagonalization algorithm to obtain a matrix P such that  $D = P^T A P$

(4 marks)

c) Consider the basis U of a substance of  $R^4$  spanned by the vectors

$\{v_1 = (1,1,1,1), v_2 = (1,1,2,4), v_3 = (1,2,-4,-3)\}$ , use the Gram Schmidt formula to find an orthonormal basis. (4 marks)

**QUESTION THREE (20 MARKS)**

a) Given that  $A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 5 & -3 & 1 \\ 1 & 6 & -2 & 4 \\ 0 & 1 & -3 & 7 \end{bmatrix}$  determine the number  $n_k$  and the sum  $S_k$  of principal minors of

order 1, 2 and 4.

(7 marks)

b) Let  $A = \begin{bmatrix} 2 & -3 & 3 \\ 3 & -4 & 3 \\ 6 & -6 & 5 \end{bmatrix}$

- i. Find the characteristic polynomial of A. (3 marks)
- ii. Find all the eigenvalues of A and their corresponding eigenvectors. (6 marks)
- iii. Is A diagonalizable? If yes, Determine the matrices P and D such that  $D = P^{-1}AP$  such that D is diagonal. (1 mark)
- iv. Find a matrix B such that  $B^2 = A$  (3 marks)

**QUESTION FOUR (20 MARKS)**

- a) State Cayley-Hamilton theorem and verify using a linear operator  $T : R^2 \rightarrow R^2$  defined by  $T(x_1, x_2) = (4x_1 - 3x_2, x_1 + 5x_2)$

(6 marks)

- b) Find the characteristic polynomial and hence the minimal polynomial of the matrix

$$A = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

(6 marks)

- i) Find an orthonormal basis for the solution space to the orthonormal system of equations below

$$x_1 + x_2 + 7x_4 = 0$$

$$2x_1 + x_2 + 2x_3 + 6x_4 = 0$$

(8 marks)

**QUESTION FIVE (20 MARKS)**

j) Let  $A = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$  be a 5-square matrix and  $I = \{1,2,4\}$  and  $J = \{1,3,5\}$  be row and

column indices of A respectively.

- i. find the minors  $A(I;J)$ ,  $A(I';J')$  and their corresponding signed minors (2 marks)
- ii. find the minimal polynomial for A (3 marks)

- k) i. Define a complex inner product space.  
(2 marks)
- ii. Let  $V$  be a complex inner product space, verify that  
 $\langle u, av_1 + bv_2 \rangle = \bar{a} \langle u, v_1 \rangle + \bar{b} \langle u, v_2 \rangle$  (2 marks)
- iii. Suppose  $\langle u, v \rangle = 3 + 2i$ , evaluate  $\langle (3 - 6i)u, (5 - 2i)v \rangle$   
(3 marks)
- l) Consider the quadratic form  $q(x, y) = 3x^2 + 2xy - y^2$  and the linear substitution  $x = s - t$  and  $y = s + t$
- i. Rewrite  $q(x, y)$  in matrix notation and find the matrix notation and find the matrix  $A$  representing  $q(x, y)$  (1 mark)
- ii. Rewrite the linear substitution using matrix notation and find the matrix  $P$  corresponding to the substitution (3 marks)
- iii. Write the quadratic form  $q(s, t)$  (2 marks)
- iv. Verify part iii above using direct substitution (2 marks)