CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

FIRST EXAMINATION FOR THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS (PURE)

MATH 802: GENERAL TOPOLOGY II

STREAMS: MSC (PURE MATHS) Y1BIII TIME: 3 HOURS

DAY/DATE: TUESDAY 06/04/2021 8.30 A.M. – 11.30 A.M.

INSTRUCTIONS:

- **Answer ANY THREE Questions.**
- **Do not write on the question paper.**

QUESTION ONE: (20 MARKS)

(a) (i) When is a topological space X said to be connected? Explain why connectedness is a topological property. (2 marks)

(b) (ii) Let Y be a subspace of X, define a separation of Y as a pair of disjoint non-empty sets A and B whose union is Y, neither of which contains a limit point of the other. Prove that the space Y is connected if there exists no separation of Y. (4 marks)

(iii) Prove that the union of a collection of connected subspaces of X that have a common point is connected. (4 marks)

(c) Define a path in a space X. Hence show that a path-connected space X is connected. (3 marks)

(d) Define a locally compact space X. Hence show that the real line R is a locally compact space whereas the set of rational numbers is not. (3 marks)

(e) Explain the basic idea behind the Stone-Cech Compactification. Under what condition does Stone-Cech Compactification coincide with Wallman Compactification? (4 marks)

QUESTION TWO: (20 MARKS)

(a) State without proof

(i) The Urysohn lemma

(ii) Urysohn Metrization Theorem

(iii) The Nagata-Smirnov metrization theorem (5 marks)

(b) State and prove the Tychonoff Theorem. (15 marks)

QUESTION THREE: (20 MARKS)

(a) Let X, Y be topological spaces. Define projection maps π_1 and π_2 such that

$\pi_1 : X \times Y \to X$ by $\pi_1(x, y) = x$ and $\pi_2 : X \times Y \to Y$ by $\pi_2(x, y) = y$.

Show that the collection $S = \{\pi_1^{-1}(U) : U \, is \, open \in X\} \cup \{\pi_1^{-1}(V) : V \, is \, open \in Y\}$ is a subbasis for the product topology on $X \times Y$. (6 marks)

(b) Suppose $d \wedge d'$ are two metrics on the set X. Let $\tau \wedge \tau'$ be the topologies the metrics induce, respectively. Show that τ' is finer than τ iff for each x in X and each $\varepsilon > 0, \exists\, a\, \delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$ (4 marks)

(c) Define the Euclidean metric d on R^n by the equation s

$$d(x, y) = \| x - y \| = \left[(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2\right]^{\frac{1}{2}}$$

and the square metric ρ by the equation $\rho(x, y) = Max\{|x_1 - y_1|, \ldots\ldots, |x_1 - y_1|\}$

Prove that the topologies on R^n induced by the Euclidean metric d and the square metric ρ are the same as the product topology on R^n. (10 marks)

QUESTION FOUR: (20 MARKS)

(a) (i) Using an appropriate illustration define the concept of a connected space X. (3 marks)

(ii) Prove that the union of a collection of connected subspaces of X that have a common point is connected. (4 marks)

(a) Define a uniform convergence sequence in a metric space X. Hence state the uniform limit theorem. (4 marks)

(b) Prove that the space R^n is paracompact. (9 marks)
