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ON UNITARY QUASI-EQUIVALENCE AND PARTIAL ISOMETRY OPERATORS IN HILBERT SPACES

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ABSTRACT

Properties of almost similarity and unitary equivalence operators on different classes of operators have been established by various researchers in the recent past. On the other hand, unitary quasi- equivalence has been shown to preserve unitary, normality, hyponormality and binormality of operators. However, properties of unitary quasi-equivalence on partial isometric operators has not been fully established. This study therefore, determines properties of unitary quasi-equivalence on partial isometric operators.

Key words: Unitary quasi-equivalence, isometry, co-isometry, partial isometry, operator and Hilbert space.

INTRODUCTION

Let H be a complex Hilbert space and (H) be a set of bounded linear operators on Hilbert space H. $A^{u.q.e}B$ and $A^{a.s}B$ denotes unitary quasi-equivalent and almost similar operators respectively. An operator $V \in (H)$ is defined as a structure preserving map. For $V \in (H, V^*$ is called the adjoint of an operator V.Two operators $A, B \in (H)$ are said to be unitarily quasi-equivalent if there exists a unitary operator $U \in B(H)$ such that the following conditions are satisfied $A^*A = UB^*BU^*AA^* = UBB^*U^*$

Othman (1996) investigated unitary quasi-equivalence under the concept of near equivalence. This class of equivalence relation operators was later studied further by Kutkut (1998). Muteti (2014) established unitary quasi-equivalence to be an equivalence relation operator. Muteti (2014), also established that unitary equivalence implies unitary quasi-equivalence. However, the converse is not necessary true unless the operators are similar normal. Thereafter Muteti (2014) determined that unitary quasi-equivalence preserve normality of operators. Nzimbi & Wanyonyi (2020) further established that projection unitary quasi-equivalent operators imply that the operators are unitary equivalent. Nzimbi & Wanyonyi (2020) also established that unitary quasi-equivalence preserves binormality and hyponormality of operators. Nzimbi & Wanyonyi (2020), also established that unitary quasi-equivalence preserves unitary operator properties. That is, if two operators are unitarily quasi-equivalent and one is a unitary operator so is the other unitary. However, the same results have not been established for isometry, co-isometry and partial isometry operators. This study therefore determines properties of unitary quasi-equivalence on isometry, co-isometry and partial isometry operators.

Definition 1.1: A Hilbert space H (Berberian, 1999). A Hilbert space H is a complete inner product space.

Definition 1.2: (Halmos, 2017). An operator $T \in (H)$ is said to be unitary if $TT^* = T^*T = I$ Isometry if $T^*T = I$.Co-isometry if $TT^* = I$.Partial isometry if T^*T is a projection or $T=TT^*T$.

From the above definitions, we have the following class inclusions: Unitary operators ⊆ isometry operators ⊆ partial isometry operators. Unitary operators ⊆ co-isometry operators ⊆ partial isometry operators

Definition 1.3: Unitary quasi-equivalence operators (Kutkut, 1998). Two operators $A, B \in (H)$ are said to be unitarily quasi-equivalent if there exists a unitary operator $U \in B(H)$ such that

$A^*A = UB^*BU^* AA^* = UBB^*U^*$

METHODOLOGY

In achieving this objective, properties of partial isometry operators are useful. In addition, properties of unitary equivalence on partial isometry operators are also used. This study aimed to extend the following Lemmas to unitary quasi-equivalence.

Lemma 2.1: (Luketero & Khalagai, 2020): Let $A \in (H)$ be a partial isometry and $B \in B(H)$ be any other operator such that, either $A = UBU^*$or $A = U^*BU$ where U is unitary, then $B \in B(H)$ is also a partial isometry.

Proof:
(I) If $A = UBU^*$ then $A^* = UB^*U^*$.
Since A is a partial isometry, then we have $A = AA^*A$.
Therefore, $A = AA^*A = UBU^*(UBU^*)^*UBU^* = UBU^*UB^*U^*UBU^* = UBB^*BU^* = UBU^*$
Thus
$UBU^* = UBB^*BU^*$ … (i)
Now pre-multiplying by U^* and post-multiplying by U in (i) above, we obtain
$B = BB^*B$.

Therefore B is also a partial isometry.
(II) Again, if $A = U^*BU$ then $A^* = U^*B^*U$
Since A is a partial isometry, we have $A = AA^*A$.
It therefore, follows that $A = AA^*A = U^*BUU^*B^*UU^*BU = U^*BB^*BU = U^*BU$
thus $U^*BB^*BU = U^*BU$ … (ii)
Pre-multiplying by U and post-multiplying by U^*equation (ii) above,
$\Rightarrow BB^*B = B$ is obtained.
Therefore, B is also a partial isometry

Lemma 2.2: (Kiprop *et al., 2021*). Let $P, Q \in (H)$ such that $P \underset{\approx}{^{a.s}} Q$. If P^2 is a partial isometry and Q is self-adjoint, then $Q^2$is also partially isometric.

Proof
Since P^2 is a partial isometry, we have $P^2 = P^2 P^{2^*} P^2$ and by projection property, we also have $PP^* = PP = P^2$.
Since $P \underset{\approx}{^{a.s}} Q$, implies there exists an invertible operator N such that
$P^*P = N^{-1}Q^*QN$ ... (i)

$P^* + P = N^{-1}(Q^* + Q)N$ ... (ii)Since Q is self-adjoint then $QQ^* = QQ = Q^2$, then (i) becomes$P^*P = N^{-1}Q^2N$.Consequently, $P^*P = P^2$ and thus $P^2 = N^{-1}Q^2$ .It follows that $Q^2 = Q^2 Q^{2^*} Q^2$, which is equivalent to $Q^2 - Q^2 Q^{2^*} Q^2 = 0$.
$\Rightarrow Q^2(1 - Q^{2^*} Q^2) = 0.$
Implying that $Q^{2^*}Q^2 = I$, or $Q^*Q =$.
Using (ii), it follows that,$(Q^* + Q)^2 = Q^{2^*} + 2Q^2 + 2Q^2 = 4Q^2$. Hence, Q^2 is a partial isometry as claimed.

MAIN RESULTS

The following are results that were established.
Theorem 3.1: If $A, B \in (H)$ are unitarily quasi-equivalent then A is isometry if and only if B is isometry.

Proof
Suppose that an operator A is isometry
Since $A \overset{u.q.e}{\approx} B$, by definition 1.3, (Kutkut, 1998), these operators satisfy the following conditions.
$A^*A = UB^*BU^*$ ... (i)

$AA^* = UBB^*U^*$ ... (ii)

But A is isometry thus by definition 1.2, Halmos, (2017).
$\Rightarrow A^*A = I$ ... (iii)

Substituting equation (i) in equation (iii)
$\Rightarrow I = A^*A = UB^*BU^*$

$\Rightarrow UB^*BU^* = I$ ... (iv)

Pre-multiplying both sides of equation (iv) with U^* and post multiplying with U we get,
$U^*UB^*BU^*U = U^*IU$

$\Rightarrow U^*UB^*BU^*U = U^*U$ Since an operator U is unitary, then by definition 1.2, Halmos (2017), Then, $U^*U = UU^* = I \Rightarrow IB^*BI = I \Rightarrow B^*B = I$Thus B is isometric. Conversely, suppose that an operator B is isometry, and then by definition 1.2,

$B^*B = I$.However, $A \overset{u.q.e}{\approx} B$ implies that equation (i) and (ii) holds. Thus substituting $B^*B = I$ in equation (i),$\Rightarrow A^*A = UB^*BU^* = UIU^*$

$\Rightarrow A^*A = UU^*$

Since an operator U is unitary then
$U^*U = UU^* = I$

$\Rightarrow UU^* = I.$

$\Rightarrow A^*A = I$

Thus by definition, an operator A is isometry.**Theorem 3.2:** Let $V, W \in (H)$ be unitarily quasi-equivalent operators, then an operator V is co-isometry if and only if$W$ is co-isometry. Proof
Since $V \overset{u.q.e}{\approx} W$, then by definition (Kutkut, 1998),
$V^*V = UW^*WU^*$ ... (i)

$VV^* = UWW^*U^*$ ... (ii)

Suppose S is co-isometry then by definition, (Halmos, 2017).
$< V^*x, V^*y > = < x, y >, \forall\, x, y \in H.$

$\Rightarrow VV^* = I$

Substituting $VV^* = I$ in equation (ii)
$\Rightarrow I = VV^* = UWW^*U^*$

$\Rightarrow UWW^*U^* = I \ldots (iii)$

Pre-multiplying both sides of equation (iii) with U^* and post multiplying with U,
$\Rightarrow U^*UWW^*U^*U = U^*IU$

$\Rightarrow U^*UWW^*U^*U = U^*U$

But $U^*U = UU^* = I$, since an operator U is unitary
$\Rightarrow IWW^*I = I$

$\Rightarrow WW^* = I.$

Thus W is co-isometry.
Conversely suppose that an operator W is co-isometry, Then by definition 1.2,
$WW^* = I$

Substituting $WW^* = I$ in (ii) implies that
$VV^* = UWW^*U^* = UIU^*$

$\Rightarrow VV^* = UIU^*$

$\Rightarrow VV^* = UU^* \Rightarrow VV^* = I$. (Since $UU^* = I$) Thus V is co-isometry.

Remark: Results 1 and 2 implies that unitary quasi-equivalence preserves isometric and co-isometric properties of operators.

Theorem 3.3: If $K, L \in (H)$ are unitarily quasi-equivalent and K is partial isometry, then an operator L is also a partial isometry operator.

Proof

Since $K \overset{u.q.e}{\approx} L$, then by definition 1.3, (Kutkut, 1998), $K^*K = UL^*LU^* \ldots (i)$ $KK^* = ULL^*U^* \ldots (ii)$ Since K is partial isometry, then by definition 1.2, Salhi & Zerovali. (2019), K^*K is a projection.

$\Rightarrow (K^*K)^2 = K^*K \ldots (iii)$
But by equation (i)
$K^*K = UL^*LU^*$

Replacing K^*K with UL^*LU^* in equation (iii)
$\Rightarrow UL^*LU^* = (UL^*LU^*)^2$
$\Rightarrow UL^*LU^* = UL^*LU^*.UL^*LU^*$

But $U^*U = I$
$\Rightarrow UL^*LU^* = UL^*LIL^*LU^*$

$\Rightarrow UL^*LU^* = UL^*LL^*LU^*$

But $L^*LL^*L = (L^*L)^2$
$\Rightarrow UL^*LU^* = (L^*L)^2U^* \ldots (iv)$
Pre multiplying both sides of equation (iv) with U^* and post multiplying with U, we get,
$\Rightarrow U^*UL^*LU^*U = U^*(L^*L)^2U^*U$
But $U^*U = I$
$IL^*LI = (L^*L)^2I$
$\Rightarrow L^*L = (L^*L)^2$
This implies that L^*L is a projection thus by definition of partial isometry definition 1.2 it means that L is partial isometry.

Theorem 3.4: let $V, W \in (H)$ be self-adjoint and unitary quasi-equivalence operators, if V^2 is partial isometry so is W^2 partial isometry. Proof
Since $V \overset{u.q.e}{\approx} W$, the following conditions hold,

$V^*V = UW^*WU^* \ldots (i)$

$VV^* = UWW^*U^* \ldots (ii)$

But V, W are self adjoint unitary quasi-equivalent operators, Nzimbi & Wanyonyi. (2020), established that V^2, W^2 are unitarily equivalent. That is by (i) and (ii) and definition of self-adjoint operator,

$V^*V = VV^* = VV = V^2$ and $W^*W = WW^* = WW = W^2$

$\Rightarrow V^2 = UW^2U^*$

Since V^2 is partial isometry then by definition of partial isometry

$V^2 = V^2 {V^2}^* V^2 \ldots (iii)$

But $V^2 = UW^2U^*$.

Replacing V^2 with UW^2U^* in equation (iii)

$\Rightarrow UW^2U^* = UW^2U^*(UW^2U^*)^*UW^2U^*$

But $(UW^2U^*)^* = UW^{2*}U^*$

$\Rightarrow UW^2U^* = UW^2U^*UW^{2*}U^*UW^2U^*$

$\Rightarrow UW^2U^* = UW^2 I W^{2*} W^2 U^*$, (since $U^*U = I$)

$\Rightarrow UW^2U^* = UW^2W^{2*}W^2U^* \ldots (iv)$

Pre-multiplying both sides of equation (iv) by U^* and post multiplying by U, We have

$U^*UW^2U^*U = U^*UW^2W^{2*}W^2U^*U$

But $U^*U = I$

$\Rightarrow IW^2I = IW^2W^{2*}W^2I$

$\Rightarrow W^2 = W^2{W^2}^*W^2 \ldots (v)$

Equation (v) implies that W^2 is a partial isometric operator.

CONCLUSION

From the results established above, we can conclude that unitary quasi-equivalent operators preserves; isometry, co- isometry and partial isometric properties. That is if two operators are unitarily quasi-equivalent and one is isometry, co- isometry or partial isometry then so is the other operator isometry, co-isometry or partial isometry respectively.

RECOMMENDATION

From the findings of this study, this results can be extended to other equivalence relation operators such as quasi-similarity and metric equivalence for further research. Similarly since the study determined properties of unitary quasi-equivalence on partial isometric operators, similar research can be done to other classes of operators such as $\theta -$ operators by utilizing their properties.

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