

Properties of Unitary Quasi-Equivalence on Isometry, Co-Isometry, and Partial Isometry Operators

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ABSTRACT

The present study aims to determine the properties of unitary quasi-equivalence and isometry, co-isometry and partial isometry operators. Unitary quasi-equivalence has been shown to be an equivalence relation. Similarly, unitary quasi-equivalence has been proven to preserve normality, hyponormality and binormality of operators. However, the properties of unitary quasi-equivalence and partial isometric operators have not been established. Based on the preceding results, it establishes that unitary quasi-equivalent operators preserve; isometry, co-isometry and partial isometric properties.

Keywords: Unitary quasi-equivalence; isometry; co-isometry; partial isometry; operator and Hilbert space.

1. INTRODUCTION

Let H be a complex Hilbert space and $B(H)$ be a set of bounded linear operators on Hilbert space H . $A \overset{u.q.e.}{\approx} B$ and $A \overset{a.s.}{\approx} B$ denotes unitary quasi-equivalent and almost similar operators respectively. An operator $V \in B(H)$ is defined as a structure preserving map. For $V \in B(H)$, V^* is called the adjoint of an operator V [1]. Two operators $A, B \in B(H)$ are said to be unitary quasi-equivalent if there exists a unitary operator $U \in B(H)$ such that the following conditions are satisfied:

$$A^*A = UB^*BU^*$$

$$AA^* = UBB^*U^*$$

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Nzimbi et al. [2] investigated unitary quasi-equivalence under the concept of near equivalence and was able to establish that, two operators $A, B \in B(H)$ are unitarily quasi- equivalent if and only if A^*A and B^*B are unitarily equivalent and AA^* and BB^* are unitarily equivalent. This class of equivalence relation operators was later studied further by Ko et al. [3]. It has been established unitary quasi-equivalence to be an equivalence relation operator by Luketero et al. [4]. Luketero et al. [4], also established that unitary equivalence implies unitary quasi-equivalence. Unless the operators are similar normal, the converse is not always true. Thereafter [5] determined that unitary quasi-equivalence preserve normality of operators [5] Further established that projection unitary quasi-equivalent operators imply that the operators are unitary equivalent. On other classes of equivalence relation such as unitary equivalence and almost similarity it has been established that they preserve partial isometric properties as established [6-9,10a]. Nzimbi et al. [5] established that unitary quasi-equivalence preserves binormality and hyponormality of operators. Nzimbi et al. [5], also established that unitary quasi-equivalence preserves unitary operator properties. That is, if two operators are unitarily quasi-equivalent and one of them is a unitary operator, the other must also be a unitary operator. However, the same results have not been established for isometry, co-isometry and partial isometry operators. As a result, the purpose of this research is to determine the properties of unitary quasi-equivalence on isometry, co-isometry, and partial isometry operators [1].

2. DEFINITIONS AND TERMINOLOGIES

Definition 2.1: A Hilbert space H [10]. A Hilbert space H is a complete inner product space [1].

Definition 2.2: (Halmos, 2017) [11]. An operator $T \in B(H)$ is said to be

- (i) Unitary if $TT^* = T^*T = I$
- (ii) Isometry if $T^*T = I$.
- (iii) Co-isometry if $TT^* = I$.
- (iv) Partial isometry if T^*T is a projection or $T=TT^*T$.

Definition 2.3: [6]. An operator $V \in B(H)$ is said to be partial isometry if [1]

$$V^2 = V^2 (V^2)^* V^2$$

As a result of the above definitions, the following class inclusions hold:

- (i) Unitary operators \subseteq isometry operators \subseteq partial isometry operators.
- (ii) Unitary operators \subseteq co-isometry operators \subseteq partial isometry operators.

Definition 2.4: Unitary quasi-equivalence operators [9]. Two operators $A, B \in B(H)$ are said to be unitarily quasi-equivalent if there exists a unitary operator $U \in B(H)$ such that:

$$A^*A = UB^*BU^*$$

$$AA^* = UBB^*U^*$$

3. METHODOLOGY

Properties of partial isometry operators were useful. For instance, definition of isometry, co-isometry and partial isometry played a vital role. In addition, properties of unitary equivalence on partial isometry operators were also used in comparison. The study aimed to extend the following Lemmas to unitary quasi-equivalence operators [1].

Lemma 3.1: [4]. Let $A \in B(H)$ be a partial isometry and $B \in B(H)$ be any other operator such that, either $A = UBU^*$ or $A = U^*BU$ where U is unitary, then $B \in B(H)$ is also a partial isometry.

Lemma 3.2: [7]. Let $P, Q \in B(H)$ such that $P \overset{a.s.}{\approx} Q$. If P^2 is a partial isometry and Q is self-adjoint, then Q^2 is also partially isometric.

Lemma 3.3: [12]. For an operator $T \in B(H)$, the following statements are equivalent [1];

- (i) T is partial isometry.
- (ii) T^* is partial isometry.
- (iii) TT^* is projection.
- (iv) T^*T is projection.
- (v) $T^*TT^* = T^*$.
- (vi) $TT^*T = T$.

4. MAIN RESULTS

The following results were established [1].

Theorem 3.1: If $A, B \in B(H)$ are unitarily quasi-equivalent then A is isometry if and only if B is isometry.

Proof: Suppose that an operator A is isometry

Since $A \overset{u.q.e.}{\approx} B$, by definition [9], the operators satisfy the following conditions.

$$A^*A = UB^*BU^* \dots (i)$$

$$AA^* = UBB^*U^* \dots (ii)$$

But A is isometry therefore by definition 2.2 [11].

$$A^*A = I$$

Substituting $A^*A = I$ in equation (i) implies that,

$$I = A^*A$$

$$= UB^*BU^*$$

Therefore,

$$UB^*BU^* = I \dots (iii)$$

Pre-multiplying both sides of equation (iii) with U^* and post multiplying with U , it implies [1]

$$U^*UB^*BU^*U = U^*IU$$

$$U^*UB^*BU^*U = U^*U \dots (iv)$$

Since an operator U is unitary, then by definition 2.2, [11] therefore, by definition,

$$U^*U = UU^*$$

$$= I$$

Equation (iv) therefore implies that,

$$IB^*BI = I$$

$$B^*B = I$$

Thus B is isometry.

Conversely, suppose that an operator B is isometry, by definition 2.2,

$$B^*B = I.$$

However, $A \overset{u.q.e}{\approx} B$ implies that equation (i) and (ii) holds [1].

Therefore substituting $B^*B = I$ in equation (i),

$$A^*A = UB^*BU^*$$

$$= UIU^*$$

$$A^*A = UU^*$$

Since an operator U is unitary then,

$$U^*U = UU^*$$

$$= I$$

Implying that,

$$UU^* = I \dots (vi)$$

Therefore equation (vi) becomes,

$$A^*A = I$$

Therefore, by definition, an operator A is isometry.

Theorem 3.2: Let $V, W \in B(H)$ be unitarily quasi-equivalent operators, then an operator V is co-isometry if and only if W is co-isometry [1].

Proof:

Since $V \overset{u.q.e}{\approx} W$, then by definition [3],

$$V^*V = UW^*WU^* \dots (i)$$

$$VV^* = UWW^*U^* \dots (ii)$$

Suppose an operator V to be co-isometry then by definition 2.2, [11]

Since an operator U is unitary, then by definition 2.2, [11] therefore, by definition,

$$U^*U = UU^*$$

$$= I$$

Equation (iv) therefore implies that,

$$IB^*BI = I$$

$$B^*B = I$$

Thus B is isometry.

Conversely, suppose that an operator B is isometry, by definition 2.2,

$$B^*B = I.$$

However, $A \overset{u.q.e}{\approx} B$ implies that equation (i) and (ii) holds.

Therefore substituting $B^*B = I$ in equation (i),

$$A^*A = UB^*BU^*$$

$$= UIU^*$$

$$A^*A = UU^*$$

Since an operator U is unitary then,

$$\begin{aligned} U^*U &= UU^* \\ &= I \end{aligned}$$

Implying that,

$$UU^* = I \dots (vi)$$

Therefore equation (vi) becomes,

$$A^*A = I$$

Therefore, by definition, an operator A is isometry.

Theorem 3.2: Let $V, W \in B(H)$ be unitarily quasi-equivalent operators, then an operator V is co-isometry if and only if W is co-isometry [1].

Proof:

Since $V \overset{u.q.e}{\approx} W$, then by definition [3],

$$V^*V = UW^*WU^* \dots (i)$$

$$VV^* = UWW^*U^* \dots (ii)$$

Suppose an operator V to be co-isometry then by definition 2.2, [11]

$$\langle V^*x, V^*y \rangle = \langle x, y \rangle, \forall x, y \in H.$$

$$VV^* = I$$

Substituting $VV^* = I$ in equation (ii)

$$I = VV^*$$

$$= UWW^*U^*$$

$$UWW^*U^* = I \dots (iii)$$

Pre-multiplying both sides of equation (iii) with U^* and post multiplying with U ,

$$U^*UWW^*U^*U = U^*IU$$

$$U^*UWW^*U^*U = U^*U \dots (iv)$$

But $U^*U = UU^* = I$, since an operator U is unitary [1]. Equation (iv) becomes

$$IWW^*I = I$$

$$WW^* = I.$$

Therefore, an operator W is co-isometry.

Conversely suppose that an operator W is co-isometry,

By definition 1.2,

$$WW^* = I$$

Substituting $WW^* = I$ in (ii) implies that, [1]

$$VV^* = UWW^*U^*$$

$$= UIU^*$$

$$VV^* = UIU^*$$

$$VV^* = UU^*$$

$$VV^* = I. \text{ (Since } UU^* = I)$$

Therefore, an operator V is co-isometry.

Remark: Results one and two implies that unitary quasi-equivalence preserves isometric and co-isometric properties of operators.

Theorem 3.3: If $K, L \in B(H)$ are unitarily quasi-equivalent and K is partial isometry, then an operator L is also a partial isometry operator [1].

Proof:

Since $K \overset{u.q.e}{\approx} L$, then by definition 1.3, [9],

$$K^*K = UL^*LU^* \dots (i)$$

$$KK^* = ULL^*U^* \dots (ii)$$

Since K is partial isometry, then by definition 2.2, [12]. K^*K is a projection. That is,

$$(K^*K)^2 = K^*K \dots (iii)$$

But by equation (i)

$$K^*K = UL^*LU^*$$

Replacing K^*K with UL^*LU^* in equation (iii) implies that,

$$UL^*LU^* = (UL^*LU^*)^2$$

$$UL^*LU^* = UL^*LU^*.UL^*LU^*$$

But $U^*U = I$ therefore,

$$UL^*LU^* = UL^*LIL^*LU^*$$

$$UL^*LU^* = UL^*LL^*LU^* \dots (iv)$$

But $L^*LL^*L = (L^*L)^2$ therefore equation (iv) becomes,

$$UL^*LU^* = U(L^*L)^2U^* \dots (v)$$

Pre multiplying both sides of equation (v) with U^* and post multiplying with U , we get,

$$U^*UL^*LU^*U = U^*U(L^*L)^2U^*U$$

But $U^*U = I$

$$IL^*LI = I(L^*L)^2I$$

$$L^*L = (L^*L)^2 \dots (vi)$$

Equation (vi) implies that L^*L is a projection. Therefore, by definition of partial isometry [12] it means that L is partial isometry [13-15,1].

Theorem 3.4: let $V, W \in B(H)$ be self-adjoint and unitary quasi-equivalence operators, if V^2 is partial isometry so is W^2 partial isometry.

Proof:

Since $V \overset{u.q.e}{\approx} W$, then following conditions hold [1],

$$V^*V = UW^*WU^* \dots (i)$$

$$VV^* = UWW^*U^* \dots (ii)$$

V, W are self-adjoint and unitary quasi-equivalent operators, [5] established that V^2, W^2 are unitarily equivalent. That is, by (i) and (ii) and definition of self-adjoint operator [16,17,10a],

$$V^*V = VV^*$$

$$\begin{aligned} &= VV \\ &= V^2 \end{aligned}$$

and

$$\begin{aligned} W^*W &= WW^* \\ &= WW \\ &= W^2 \end{aligned}$$

Implying that,

$$V^2 = UW^2U^*$$

Since V^2 is partial isometry then by definition of partial isometry [1, 6]

$$V^2 = V^2V^2 \quad V^2 \dots \text{(iii)}$$

But $V^2 = UW^2U^*$

Therefore, replacing V^2 with UW^2U^* in equation (iii)

$$UW^2U^* = UW^2U^*(UW^2U^*)^*UW^2U^*$$

But $(UW^2U^*)^* = UW^2U^*$ therefore,

$$UW^2U^* = UW^2U^*UW^2U^*UW^2U^*$$

$$UW^2U^* = UW^2IW^{2*}IW^2U^*, \text{ (since } U^*U = I)$$

Therefore,

$$UW^2U^* = UW^2W^2W^2U^* \dots \text{(iv)}$$

Pre-multiplying both sides of equation (iv) by U^* and post multiplying by U , [1]

We have

$$U^*UW^2U^*U = U^*UW^2W^2W^2U^*U$$

But $U^*U = I$ therefore,

$$IW^2I = IW^2W^2W^2I$$

$$W^2 = W^2W^2W^2 \dots \text{(v)}$$

Equation (v) implies that W^2 is a partial isometric operator.

Remark: The results established above can be applied in algebra and differential operators useful for calculation and differentiation of wave functions. The theorems can also be useful in knowledge advancement on interpretation of equivalence relation of operators in Hilbert spaces.

5. CONCLUSION

Based on the preceding results, it establishes that unitary quasi-equivalent operators preserve; isometry, co-isometry and partial isometric properties [1]. That is, if two operators are unitarily quasi-equivalent and one of them is isometry, co-isometry, or partial isometry, then the other operator is also isometry, co-isometry, or partial isometry.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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Biography of author(s)



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- ✓ 2022-2026- Hypercyclicity of Elementary Operators and the Hypercyclicity Criterion-(HelB) Scholarship-PhD Supervision for Kawira Esther
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- ✓ Transmission -Chuka University College Staff Research Grant-(2012)
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- ✓ **Norm Estimates for Norm- Attainable Operators.**-(Nacosti) Research Grant -PhD Research Jerimiah Ndung'u Kinyanjui.

Any other remarkable point(s): He is supervising 2 PhD students who are expected to graduate in 2025. He is also guiding 2 MSc students to complete and now guiding 4 students.

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