



Correspondence of Fixed-Point Theorem in $T_2, T_3 - SPACE$

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

Fixed-point theory (FPT) has lot of applications not only in the field of mathematics but also in various other disciplines. Fixed Point Theorem presents that if $T: X \rightarrow X$ is a contraction mapping on a complete metric space (X, d) then there exists a unique fixed point in X . FPT is also essential in game theory, in this case Brower Fixed Point has an application in game theory specifically in non-cooperative games and existence of Equilibrium. In particular, a game is a set of actions done by the participants defined by a set of rules. This is commonly described using mathematical concepts, which offers a concrete model to describe a variety of situations. On the other hand, the separation axioms $T_i, i = 0,1,2,3,4$ are vital properties that describes the topological spaces T_0, T_1, T_2, T_3 and T_4 . It is noted that a $T_3 - space$ is a generalized version of T_2 -space and since various results on application of fixed point theory in game theory on an arbitrary locally convex $T_2 - space$ has been established, in this study we sort to extend this concept to the general $T_3 - space$. The utilization of a symmetric property of Hausdorff space established that if two continuous commutative mappings are defined on a $T_3 - space$, then the two maps achieves unique fixed points.

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1 Introduction

A topological space X is said to be a Hausdorff space (T_2 - space) if $\forall a, b \in X, a \neq b$, then $\exists$ an open sets U and V such that $a \in U, b \in V$, and $U \cap V = \emptyset$, whereas a T_3 - space is a regular T_1 - space (Kuratowski, K. [1]. Furthermore, a topological space X is regular if $\forall a \in X$ and any closed set A of X , $\exists$ open sets say U and V such that $a \in U, A \subseteq V$ and $U \cap V = \emptyset$, while a topological space X is said to be T_1 - space if for any two points $a, b \in X$, where $a \neq b$ there exist open sets U and V such that $a \in U, b \notin U$ and $b \in V, a \notin V$ (Thron, W. J. [2]. In this study, we consider T_i - spaces for $i = 0, 1, 2, 3, 4$ which are key separable topological spaces. In this case, $T_4 \rightarrow T_3 \rightarrow T_2 \rightarrow T_1 \rightarrow T_0$. A case of interest in our study is the result that a T_3 - space implies T_2 - space as illustrated by the following lemma.

Lemma 1.1: Every topological space T_3 - space is a T_2 - space Munkres et al., [3].

We also present of an important property of a symmetric Hausdorff space as presented by Gupta, V., Aydi, H., & Mani, N. [4] whereby, it is illustrated that a Hausdorff space X with a continuous mapping H is said to be a symmetric if it satisfies the following axioms;

- I. $H(x, y) = 0$ iff $x = y$
- II. $H(x, y) = H(y, x)$
- III. It is a T_1 - space

Having seen that T_3 - space is a generalization of T_2 - space and suppose that the T_3 - space satisfies a symmetric conditions, then as a consequence of Lemma 1.1 the following result is evident,

Lemma 1.2: Let X be a T_3 - space and $R: X \rightarrow X$ be a continuous mapping such $\forall x, y \in X$, then

- I. $R(x, y) = 0$ iff $x = y$.
- II. It is a regular T_1 - space
- III. $R(x, y) = R(y, x)$

Thus R is said to be symmetric on T_3 - space.

From lemma 1.1 and lemma 1.2 we have established T_2 - space and T_3 - space are symmetric .

Popa, [5] established a generalized results of Banach Fixed point theorem through Hausdorff topological space by considering the properties below:

1. $H(x, y) \neq 0$ (1)
2. $H(Fx, Fy) \leq \alpha M(x, y) + \beta H(x, y)$ (2)
3. $H(Fx, Fy) \leq \alpha \left(\frac{H(x, Fx)H(y, Fy)}{H(x, y)} + \beta H(x, y) \right)$..(triangular inequality)..... (3)
4. $M(x, y) = \max \left\{ H(x, y), \frac{H(x, Fx)H(y, Fy)}{H(x, y)} \right\}$ (4)
5. $H^2(x, y) \geq H(x, x)H(y, y)$ (5)

Where $\alpha, \beta > 0$ and $\alpha + \beta < 1$ for some $x_0 \in X$ such that $Fx_0 = x_1$. By defining the sequence x_n in a Hausdorff space X such that $Fx_n = x_{n+1}$ and $Fx_{n-1} = x_n$, then through iteration process the sequence $x_n = \{F^n x_0\}$ has a convergent subsequent. As a result fixed point of F is attain.

Based on the results discussed above by Popa, [5] and utilizing the concept of symmetric Hausdorff space, we similarly make an effort of derivation of Fixed Point Theorem under T_3 - space in the proceeding proposition [6,7].

2 Main Results

Proposition 2.1:

Let $T: X \rightarrow X$ be a continuous mapping on T_3 - space X into itself and let $R: X \rightarrow X$ be a continuous mapping which commutes with T satisfying the following conditions [8].

1. $R(x) \subset T(x) \forall x \in X$
2. $d(Rx, Ry) \leq \alpha(Tx, Ty), \forall x, y \in X,$

Then T and R have a fixed point.

Proof:

Choosing $x_0 \in X$ such that $Rx_0 = Tx_1$. Based on this, we define a sequence x_n in X such that $Tx_n = R(x_{n-1})$

Step 1:

Let $x_0 \in X$ and x_1 be such that $Tx_1 = Rx_0$, in general we choose x_n so that $Tx_n = R(x_{n-1})$
 $T(x_{n+1}, x_n) \leq \alpha(Tx_n, Tx_{n-1})$ For all n

It follows from property (2)

$$T(x_n, x_{n+1}) \leq T(Rx_{n-1}, Rx_n) \leq \alpha T(x_{n-1}, x_n) + \beta T(x_{n-1}, x_n) < T(x_{n-1}, x_n) \dots (1)$$

Then from property (4)

$$T(x_{n-1}, x_n) = \max \{T(x_{n-1}, x_n), \frac{T(x_{n-1}, Rx_{n-1})T(x_n, Rx_n)}{T(x_{n-1}, x_n)}\}$$

$$= \max\{T(x_{n-1}, x_n), T(x_n, x_{n+1})\}$$

Suppose that

$$T(x_n, x_{n+1}) > T(x_{n-1}, x_n), \text{ it follows from equation (1)}$$

$$T(x_n, x_{n+1}) \leq \alpha T(x_{n-1}, x_n) + \beta T(x_{n-1}, x_n) \dots (2)$$

Also if

$$T(x_n, x_{n+1}) \leq T(x_{n-1}, x_n)$$

It follows again from (I)

$$T(x_n, x_{n+1}) \leq (\alpha + \beta)T(x_{n-1}, x_n) < T(x_{n-1}, x_n) \dots (3)$$

From (2) and (3) we obtain

$$\alpha T(x_n, x_{n+1}) + \beta T(x_{n-1}, x_n) \leq (\alpha + \beta)T(x_{n-1}, x_n)$$

$$\leq \alpha T(x_{n-1}, x_n) + \beta T(x_{n-1}, x_n)$$

$$\alpha T(x_n, x_{n+1}) + \beta T(x_{n-1}, x_n) \leq \alpha T(x_{n-1}, x_n) + \beta T(x_{n-1}, x_n)$$

$$\alpha T(x_n, x_{n+1}) \leq \alpha T(x_{n-1}, x_n)$$

$$T(x_n, x_{n+1}) \leq T(x_{n-1}, x_n) < T(x_{n-1}, x_n) \dots (4)$$

Using equation (3) and (4), and Repeating the process n times then,

$$T(x_n, x_{n+1}) < T(x_{n-1}, x_n) < \dots < T(x_1, x_0) < T(x_0, x_1)$$

Since $\lim_{n \rightarrow \infty} x_n$ is bounded and letting $x_n \forall n \in N$ be a convergent sequence, denote the limit by t where $t \in X$. Let $x_{nk} \forall n \in N$ be subsequence such that if $\varepsilon > 0$ then by definition of convergence for x_n as $n \in N$, there

exists $N \in \mathbb{N}$ such that $|x_n - t| < \varepsilon$ for $n \geq N$, but this value N will also work for x_{nk} , this is because if $n \geq N$ then $x_{nk} = x_m$ for some $m \geq n \geq N$ and so $|x_{nk} - t| = |x_m - t| < \varepsilon$. Thus $|x_{nk} - t| < \varepsilon$, as $n \rightarrow \infty$, $x_{nk} = t$

Hence, we obtain a monotone sequence which converge with all its subsequence to some real number $t \in X$.

Step 2

Next is to show t is a fixed point for T and R

Where $t \in X$ such that $Tx_n \rightarrow t, Rx_n \rightarrow t$

Since T is continuous it implies R is also continuous.

Since T and R commutes it follows that

$R(T(x_n)) \rightarrow R(t)$ and $T(R(x_n)) \rightarrow T(t)$ So that

$R(T(x_n)) = T(R(x_n))$

then

$T(t) = R(t)$

And

$T(T(t)) = T(R(t)) = R(R(t))$ (by commutativity)

From contraction mapping

$d(R(t), R(R(t))) \leq \alpha d(T(t), T(R(t))) = \alpha d(R(t), R(R(t)))$

$d(R(t), R(R(t))) \leq \alpha d(T(t), T(R(t)))$

$d(R(t), R(R(t))) - \alpha d(R(t), R(R(t))) \leq 0$, Since $\alpha \in (0, 1)$

$d(R(t), R(R(t)))(1 - \alpha) \leq 0$

$R(t) = R(R(t)) = T(R(t))$, then $R(t)$ is a fixed point for T and R

Step3

To show that R and T have unique fixed point, we

Suppose $R(t) = T(t) = t$ and $R(t') = T(t') = t'$

Then it follows from contraction principle,

$d(t, t') = d(R(t), R(t')) \leq \alpha d(T(t), T(t')) = \alpha d(t, t')$

but $\alpha < 1$

And thus

$t = t'$

Thus R and T have unique fixed point ■

3 Conclusion

In this study, it has been established that the generalized result by Popa, [5] of Banach Fixed point theorem in a T_2 topological space can be extended to a T_3 space if the considered T_3 - space possesses two continuous maps that commutates with one another.

Disclaimer (Artificial intelligence)

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Competing Interests

Authors have declared that no competing interests exist.

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