CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

EXAMINATION FOR THE AWARD OF DEGREE OF BACHELORS OF SCIENCE GENERAL, MATHEMATICS, EDUCATION ARTS & SCIENCE, ECONOMICS & STATISTICS, BACHELOR OF ARTS ECON-MATHS, BACHELOR OF ARTS GENERAL

MATH 401: TOPOLOGY I

STREAM: "AS ABOVE" **2 HOURS**

DAY/DATE: TUESDAY 17/12/2024 **11.30 AM -1.30 PM**

INSTRUCTIONS:

- **Answer question ONE and Any other Two questions**
- **Do not write on the question paper**

QUESTION ONE: (30 MARKS)

(a) Distinguish between a discrete topology and the usual Euclidean topology. Illustrate this using appropriate examples in each case (4 marks)

(b) *Prove that the intersection* $\tau_1 \cap \tau_2$ *of any two topologies* τ_1 *and* τ_2 *on a non-empty set X is also a topology on X.* *(5 marks)*

(c) Show that if X be a discrete topological space and that $A \subset X$, then the derived set of A, $A' = \emptyset$. (4 marks)

(d) Let $f: x_1 \to x_2$ *where* $x_1 = x_2 = \{0, 1\}$ and are such that (x_1, D) *and* $(x_2, \$)$ be defined by $f(1) = 1$ *and* $f(0) = 0$. Show that f is not continous but f^{-1} is continuous. (3 marks)

(e) (i) Define a dense set in X. (1 mark)

(ii) Hence show that if A is dense in X then for every open set $O \subset X$, $O \cap A \neq \emptyset$ (3 marks)

(f) Define a Hausdorff space. Hence prove that the set $X = \{a, b\}$ with the discrete topology is a Hausdorff space whereas the same set with the Sierpinski topology is not a Hausdorff space (5 marks)

(g) Let B be a basis for a topology T on X. Prove that if each basis element B ∈ B is open in X, so B ⊂ T (2 marks)

(h) Distinguish between a closed map and open map. Hence using a counter example show that an open function or a closed function need not be continuous (3 marks)

QUESTION TWO: (20 MARKS)

(a) Let X be an infinite set and τ be a family of subsets of X which includes ∅ and all the subsets of X for which A^c is finite, where $A^c = X\setminus A$. Then τ is a topology on X.

(8 marks)

(b) Let A be a subset of a topological space(X, τ). Prove that a subset $A\subset X$ is closed if and only if set of its derived points $A'\subset A$ (8 marks)

(c) Let (X, τ) be a topological space and $A, B\subset X$. Prove that $\overline{A\cup B} = \overline{A} \cup \overline{B}$

(4 marks)

QUESTION THREE: (20 MARKS)

(a) Consider the following topology on $X = \{p, q, r, s, t\}$ and $\tau = \{\{p\}, \{p, q\}, \{p, r, s\}, \{p, q, r, s\}, \{p, q, t\}, X, \emptyset\}$. If $B = \{p, q, r\}$. Find

(i) The exterior of B (3 marks)

(ii) The boundary of B (2 marks)

(iii) Hence show that the boundary of B, $\delta B = \overline{B} \cap \overline{X/B}$ (3 marks)

(b) Let X be a non-void set. A collection B of subsets of X is a base for a topology on X if, and only if;

i) $\bigcup\{B: B\subset B\} = X$.

ii) If $B_1, B_2 \in B$, then $B_1 \cap B_2$ is a union of members of B. Equivalently, for every $x\in B_1\cap B_2$ there is a set $B_X\in B$ such that $x\in B_X \subseteq B_1 \cap B_2$. (12 marks)

QUESTION FOUR: (20 MARKS)

(a) Let $P: X\to Y$ be an open map and let $S\subset Y$ be any subset of Y and A is a closed set in X such that $P^{-1}(S)\subset A$. Show that $S\subset B$ and $P^{-1}(B)\subset A$. (5 marks)

(b) Let $f: X \rightarrow Y$ be a bijective. Prove that the following statements are equivalent.

(i) f is a homomorphism

(ii) f is open

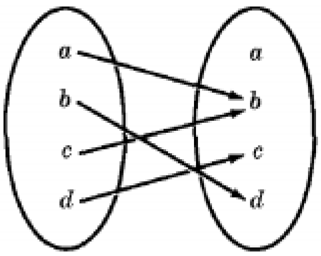
(iii) f is closed

(iv) $f(\overline{A}) = \overline{f(A)}$ *for any* $A \subseteq X$. (10 marks)

(c) Consider the following topology defined on $X = \{a, b, c, d\}$

$\tau = \{\{a, b\}, \{a\}, \{b\}, \{b, c, d\}, X, \emptyset\}$

Let the function $f: X \rightarrow X$ be defined by the diagram below



Show that f is continuous at d but not continuous at c. (5 marks)

QUESTION FIVE: (20 MARKS)

(a) Every metric space (X, τ) is a Hausdorff space with respect to its metric topology.

(5 marks)

(b) Distinguish between a $a\ T_1$ *and* T_2 space. Using an appropriate counter example show that a T_2 space $\Rightarrow T_1$ space but a T_1 space $\not\Rightarrow T_2$. (5 marks)

(c) Prove that a topological space X is a T_1 space if and only every singleton subset $\{p\} \subset X$ is closed. (5 marks)

(d) Define a local base for a topological space X. Hence prove that a point $p \in X$ is an accumulation point of $A \subset X$ if and only if every member of some local base β_p at the point p contains a point of A different from p. (5 marks)
