

**NORM OF ELEMENTARY OPERATOR IN TENSOR PRODUCT OF
 C^* -ALGEBRAS**

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**A Thesis Submitted to the Graduate School in Partial Fulfillment of the Requirements
for the Award of the Degree of Doctor of Philosophy in Pure Mathematics of Chuka
University**

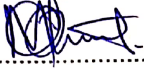
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OCTOBER, 2024

DECLARATION AND RECOMMENDATION

Declaration

This thesis is my original work and has not been presented for a degree in any other university.

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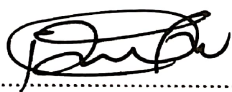
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Recommendation

This thesis has been examined, passed and submitted with our approval as university supervisors.

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DEDICATION

I dedicate this project to my lovely wife Angela and my children Lucy and Addilyn.

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First, I thank the Almighty God for the good health and guidance that He gave me all through this study. I am deeply grateful for the continuous support, insight, and patience of my supervisors, Dr. King'ang'i and Prof. Musundi. Without their constant trust and sometimes, gentle prodding, this thesis would not have been completed.

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ABSTRACT

Many properties of elementary operators, including spectrum, numerical ranges, compactness, rank, and norm have been studied in depth with the norm property attracting many researchers due to its wide range of applications. Generally, the calculation of norms involves finding a formula that describes the norms in terms of their coefficients. The discussion on the norm of the elementary operator can be traced to Stampfli's theorem of 1970 which created a proper base for the study of norms. The determination of the norm of the elementary operator in C^* -algebras, JB^* -algebras, standard operator algebras, Cartan factor, prime C^* -algebras, two-dimensional complex Hilbert space and tensor product have been studied and based on earlier research, the norm of basic elementary operator in tensor product of C^* -algebras have been evaluated using the concept of finite rank operator and Stampfli's maximal numerical range. The norms of other types of elementary operators in tensor product of C^* -algebras was not determined. This study extended the study of the norm of elementary operator in tensor product of C^* -algebras to the general finite length elementary operator. The study determined the norm of finite length elementary operator in tensor product of C^* -algebras and found for a finite length elementary operator, T_n , in a tensor product of C^* -algebras $\|T_n \setminus B(H \otimes K)\| = \sum_{i=1}^n \|A_i\| \|B_i\| \|C_i\| \|D_i\|$, $A_i, C_i \in B(H)$ and $B_i, D_i \in B(K)$. Consequently, the conditions under which the norm of an arbitrary finite length elementary operator in tensor products of C^* -algebras as is expressible in terms of the norms of its coefficient operators were also established and found that $\|T_n\| = \sum_{i=1}^n \|A_i\| \|B_i\| \|C_i\| \|D_i\|$ if $\|A_i \otimes B_i\| \in W_o(A_i \otimes B_i)$ and $\|C_i \otimes D_i\| \in W_o(C_i \otimes D_i) \forall i = 1, 2, \dots, n$. Finally, the research determined the norm of Jordan elementary operator in the tensor products of C^* -algebras where for every tensor product $X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$, then $\|U_{A \otimes B, C \otimes D}\| = 2\|A\| \|B\| \|C\| \|D\|$, $A, C \in B(H)$, $B, D \in B(K)$ and $U_{A \otimes B, C \otimes D}$ is the Jordan elementary operator in a tensor product of C^* -algebras. The techniques of rank one operator, finite rank operator, the definition of the elementary operator in tensor product, properties of tensor product, inner product, properties of functionals, and the norm were used to achieve the objectives of this study. The finding of this research can be used in areas of functional analysis, linear algebra, operator theory and mathematical physics by utilizing various properties of an elementary operator under the study of C^* -algebra.

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INDEX OF NOTATIONS

$B(H)$	Banach algebra of bounded linear operators on Hilbert space H	1
\forall	For all / For every	2
$B(H \otimes K)$	Set of bounded linear operators on Hilbert space $H \otimes K$	3
$\ \cdot\ $	Norm	9
$\langle \cdot, \cdot \rangle$	Inner product	10
ϑ	Algebra	10
ϱ	Normed Algebra	10
$M_n(\mathbb{C})$	Set of $n \times n$ matrices in complex space	11
$M(\vartheta)$	Multiplier algebra	11
$H \otimes K$	Tensor Product of Hilbert spaces H and K	12
$W(T)$	Numerical range of an operator T	12
$w(T)$	Numerical radius of T	13
$W_0(T)$	Maximal numerical range of T	13
$W_{Alg}(T)$	Algebraic numerical range of T	13
$W_{spec}(T)$	Spatial numerical range of T	13
$tgm(X, Y)$	Tracial geometry mean of X and Y	13
$S(A)$	States of a complex unital Banach algebra Ω with identity	13
$\sigma(T)$	Spectrum of an operator T	14
$r(T)$	Spectral radius of an operator T	14
$\rho(T)$	Resolvent set of an operator T	14
max	Maximum	18
$M(A)_B$	Maximal spatial numerical range of A relative to B	20
Sup	Supremum	23
Λ	Prime C*-algebra	25
$symm$	Symmetric	26

A^* The adjoint of an operator A 26

CHAPTER ONE

INTRODUCTION

1.1 Background Information

In 1884, Sylvester published a series of notes in which he calculated the Eigen values of the matrix operators corresponding to elementary operators on matrices as presented by Bleacher (1988). Later in 1959, Lumer and Rosenblum introduced the concept of the elementary operator within a broader setting of Banach algebra. The spectra of such operators were computed by Lumer and Rosenblum (1959), who placed a strong emphasis on applying these spectra to the operator equation systems.

Since then, extensive research has been done on a variety of elementary operator properties, yielding findings and expositions on several facets of elementary operators. These features include spectrum, ranges, compactness, rank, and norm. The study of norm of elementary operator can be traced back to Stampfli's theorem (1970 [Theorem 5]) which used properties of a numerical range. Numerous researchers have attempted to calculate this norm which generally involves finding a formula that describes the norms in terms of their coefficients.

Previous studies have shown determination of norm of elementary operator in C^* -algebras, JB^* -algebra, standard operator algebra, Cartan factor, prime C^* -algebra, two dimensional complex Hilbert space and tensor product. Because C^* -algebras have so many applications, many researchers have been drawn to the study of the norm of elementary operator in C^* -algebras.

Timoney (2001) determined the general norm of basic elementary operator in C^* -algebra and obtained that if \mathcal{A} is C^* -algebra, then;

$$\|M_{A,B}\| = \sup\{\|M_{A,B}(X)\| : X \in B(H)\} = \sup\{\|AXB\| : X \in B(H)\}$$

where X denotes the set of unitaries in $B(H)$.

The work by Timoney (2001) established the foundation for calculating the upper bound of the norm of the Jordan elementary operator and the finite length elementary operator in the tensor product of C^* -algebras in this study.

Okelo and Agure (2011) used the finite rank operators to determine the norm of the basic elementary operator in C^* -algebra by letting H be a Hilbert space, and $B(H)$ the algebra of

bounded linear operators on H . If $M_{A,B} : B(H) \rightarrow B(H)$ is defined by $M_{A,B} = AXB$, $\forall X \in B(H)$ where A and B are fixed in $B(H)$ then;

$$\|M_{A,B}\| = \|A\|\|B\|, \text{ for } \|X\| = 1 \text{ where } X(x) = x, \forall \text{ unit vectors } x \in H.$$

King'ang'i *et al.* (2014) extended the work of Okelo and Agure (2011) and determined the norm of elementary operator of length two for finite-dimensional separable Hilbert space $W \in B(H)$ with $\|W\| = 1$ and $W(x) = x$ for all unit vectors $x \in H$ and proved that if H is a complex Hilbert space and $B(H)$ be algebra of all bounded linear operators on H . Let E_2 be the elementary operator of length two on $B(H)$. If for an operator $W \in B(H)$ with $\|W\| = 1$, we have $W(x) = x$ for all unit vectors $x \in H$, then;

$$\|E_2\| = \sum_{i=1}^2 \|A_i\|\|B_i\|.$$

King'ang'i (2017) employed the concept of the maximal numerical range of T^*S relative to S to determine the lower bound of the norm of an elementary operator of length two by stating that if E_2 be an elementary operator of length two on $B(H)$. Then;

$$\|E_2\| \geq \sup_{\lambda \in W_B(A^*B)} \left\| \|B_1\|A_1 + \frac{\bar{\lambda}}{\|B_1\|}A_2 \right\|.$$

King'ang'i (2017) also determined the conditions under which the norm of an elementary operator of length two is expressible in terms of the norms of its coefficients operators by letting H be a complex Hilbert space and A_i, B_i be bounded linear operators on H for $i = 1, 2$. Let $0 \in W_{B_1}(B_1^*B_2) \cup W_{B_2}(B_1^*B_2)$. Then;

$$\|E_2\| \geq \|A_1\|\|B_1\|,$$

where E_2 is the elementary operator of length two.

Additionally, King'ang'i (2017) proved that if H is a complex Hilbert space and A_i, B_i be bounded linear operators on H for $i = 1, 2$. Let E_2 be an elementary operator of length two. If $\|A_1\|\|A_2\| \in W_{A_1}(A_2A_1^*)$ and $\|B_1\|\|B_2\| \in W_{B_2}(B_1^*B_2)$, then;

$$\|E_2\| = \sum_{i=1}^n \|A_i\|\|B_i\|.$$

Kawira *et al.* (2018) extended the work of King'ang'i *et al.* (2014) to the norm of finite length elementary operator and determined the norm of an elementary operator of an arbitrary length in C^* -algebra using the concept of finite rank operators and proved that for a complex Hilbert space H and $B(H)$ be the algebra of bounded linear operators on H . Let E_n be elementary operator on $B(H)$. If $\forall X \in B(H)$ with $\|X\| = 1$, we have $X(f) = f$ for all unit vectors $f \in H$ then;

$$\|E_n\| = \sum_{i=1}^n \|A_i\| \|B_i\|, n \in \mathbb{N}.$$

In 2018, Muiruri *et al.* defined the elementary operator in a tensor product of C^* -algebra as follows. Let $H \otimes K$ be tensor product of complex Hilbert spaces H and K and $B(H \otimes K)$ the set of bounded linear operators on $H \otimes K$. Then we have the following definitions:-

(a) *An elementary operator, $T_n : B(H \otimes K) \rightarrow B(H \otimes K)$, is defined as;*

$$T_n(X \otimes Y) = \sum_{i=1}^n A_i \otimes B_i(X \otimes Y)C_i \otimes D_i, \forall X \otimes Y \in B(H \otimes K),$$

$A_i \otimes B_i, C_i \otimes D_i$ being fixed elements of $B(H \otimes K)$, where $A_i, C_i \in B(H)$, the set of bounded linear operators on H and $B_i, D_i \in B(K)$, the set of bounded linear operators on K .

(b) When $n = 1$ we obtain *the basic elementary operator, $M_{A \otimes B, C \otimes D} : B(H \otimes K) \rightarrow B(H \otimes K)$, defined as;*

$$M_{A \otimes B, C \otimes D}(X \otimes Y) = A \otimes B(X \otimes Y)C \otimes D, \forall X \otimes Y \in B(H \otimes K).$$

(c) When $n = 2$ then we obtain *an elementary operator of length two* which is defined by;

$$T_2(X \otimes Y) = A_1 \otimes B_1(X \otimes Y)C_1 \otimes D_1 + A_2 \otimes B_2(X \otimes Y)C_2 \otimes D_2, \\ \forall X \otimes Y \in B(H \otimes K)$$

and $A_i \otimes B_i, C_i \otimes D_i$ being fixed elements of $B(H \otimes K)$, where $A_i, C_i \in B(H)$, the set of bounded linear operators on H and $B_i, D_i \in B(K)$, the set of bounded linear operators on K .

(d) The Jordan elementary operator, $U_{A \otimes B, C \otimes D} : B(H \otimes K) \rightarrow B(H \otimes K)$, is defined as;

$$U_{A \otimes B, C \otimes D}(X \otimes Y) = A \otimes B(X \otimes Y)C \otimes D + C \otimes D(X \otimes Y)A \otimes B, \\ \forall X \otimes Y \in B(H \otimes K),$$

$A \otimes B, C \otimes D$ being fixed elements of $B(H \otimes K)$, where $A, C \in B(H)$, the set of bounded linear operators on H and $B, D \in B(K)$, the set of bounded linear operators on K .

The definitions of Muiruri *et al.* (2018) resulted in the determination of the norm of basic elementary operator in a tensor product of C^* -algebras using different methods. Using the rank one operator and properties of tensor product Muiruri *et al.* (2018), proved that for complex Hilbert spaces H and K and $B(H \otimes K)$ be the set of bounded linear operators on $H \otimes K$. Then $\forall X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$, we have;

$$\|M_{A \otimes B, C \otimes D}\| = \|A\| \|B\| \|C\| \|D\|,$$

where A, C and B, D are fixed elements in $B(H)$ and $B(K)$ respectively.

As a consequence, Muiruri *et al.* (2018) related the norm of basic elementary operator in tensor product and the usual norm in this elementary operator in C^* -algebra and showed that if H and K are complex Hilbert spaces and $B(H \otimes K)$ be the set of bounded linear operators on $H \otimes K$. Then $\forall X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$, we have;

$$\|M_{A \otimes B, C \otimes D}\| = \|M_{A, C}\| \|M_{B, D}\|$$

, where $M_{A, C}$ and $M_{B, D}$ are basic elementary operators in $B(H)$ and $B(K)$ respectively.

Daniel *et al.* (2022) used the Stampfli's maximal numerical range to determine the norm of basic elementary operator in a tensor product and they obtained that by Letting H and K be Hilbert spaces and let $O_{A \otimes B, C \otimes D}$ be basic elementary operator on $B(H \otimes K)$. If $\forall X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$, $A, C \in B(H)$, $B, D \in B(K)$, $\zeta \in W_0(C)$, $\xi \in W_0(D)$ then we have;

$$\|O_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| = \text{Sup}_{\zeta \in W_0(C)} \text{Sup}_{\xi \in W_0(D)} \{ \|\zeta\| \|\xi\| \|A\| \|B\| \}.$$

Additionally, Daniel *et al.*, (2023) determined the bounds of the norm of elementary operator of length two in tensor product using the Stampfli maximal numerical range and proved that if H and K are Hilbert spaces and let $O_{A \otimes B, C \otimes D}$ be the elementary operator of length two on $B(H \otimes K)$. If $\forall X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$, $A_i, C_i \in B(H)$, $B_i, D_i \in B(K)$, $\zeta_i \in W_0(C_i)$, $\xi_i \in W_0(D_i) \forall i = 1, 2$ then we have;

$$\| \sum_{i=1}^n O_{2A_i \otimes B_i, C_i \otimes D_i} \setminus B(H \otimes K) \| = \text{Sup}_{\zeta_i \in W_0(C_i)} \text{Sup}_{\xi_i \in W_0(D_i)} \{ \sum_{i=1}^n |\xi_i \zeta_i| \|A_i\| \|B_i\| \}$$

Therefore this study extended the work of Muiruri *et al.* (2018) by determining the norms of finite length elementary operator and Jordan elementary operator in tensor product of C^* -algebras.

1.2 Statement of the Problem

The norm property has been a subject of interest for researchers more so in different examples of elementary operators in C^* -algebra. The norm of the elementary operator in the tensor product of C^* -algebras could have equally significant results. For instance, based on previous studies, the notion of finite rank operator and Stampfli maximal numerical range have been used to evaluate the norm of basic elementary operators in tensor products of C^* -algebras. However, the norms of other kinds of elementary operators in tensor products of C^* -algebras remain largely unknown. This study extended the determination of norm of basic elementary operator in the tensor product of C^* -algebras to the norm of finite length elementary operator and Jordan elementary operator in tensor product of C^* -algebras. Furthermore, the study determined the conditions under which the norm of an arbitrary finite length elementary operator in tensor product of C^* -algebras is expressible in terms of norms of its coefficient operators.

1.3 Objectives of the Study

1.3.1 Main Objective

The main objective of this study was to determine the norm of elementary operator in the tensor product of C^* -algebras.

1.3.2 Specific Objectives

The specific objectives of this study were to:

- (i) Determine the norm of finite length elementary operator in tensor product of C^* -algebras.
- (ii) Determine the norm of Jordan elementary operator in tensor product of C^* -algebras.
- (iii) Establish the conditions under which the norm of an arbitrary finite length elementary operator in tensor product of C^* -algebras is expressible in terms of norms of its coefficient operators.

1.4 Significance of the Study

The results of this work can be used in the following areas.

Functional analysis

Functional analysis is a branch of mathematics that deals with function spaces and their operators. The findings of the study in tensor product of C^* -algebras can contribute significantly to this field by providing a framework for studying operator algebras on tensor product of Hilbert spaces. This connection will enrich both disciplines, allowing for the development of theories like spectral theory, which is crucial for solving differential equations in physics.

Linear algebra

The results of this study being findings in elementary operators is important since elementary operators are fundamental in linear algebra as they represent basic transformations such as scaling, reflection, rotation and shearing. Thus the wider understanding of elementary operators in this study can help in studying vector spaces, linear transformations and matrix operations.

Operator theory

The findings in this study can also be central in operator theory, which deals with linear operators on function spaces and has applications in differential equations, control theory, signal processing and quantum computing.

Mathematical physics

The results in elementary operator in this study can extensively be used in mathematical physics to model physical phenomena and solve differential equations since it provides a

framework for analyzing and understanding the behavior of physical systems in various domains. It can also play a role in the study of quantum statistical mechanics and quantum probability theory since it enhances the study of C^* -algebras.

Algebraic topology

In algebraic topology, tensor products are utilized in the context of homology and cohomology theories. For instance the results of this study can be used in enhancing the study of tensor product of chain complexes which is used to define the tensor product of homology groups, providing deeper insights into the topology properties of spaces.

The representation theory

The results of this study can have immediate application in the representation theory of restricted direct product groups (such as adèle groups), since the group C^* -algebra of such a group is an infinite tensor product of the C^* -algebras of the coordinate groups. The results of this study also being a study in tensor product can help in mathematics, particularly in group theory and lie theory, since tensor products plays a vital role in the study of representations of groups and lie algebras. Thus the study can help in understanding the structure and behavior of symmetry groups and their associated algebraic structures.

Quantum mechanics

The results of this study can also serve as the mathematical foundation for describing physical systems and their dynamics. The algebra of observables and states in quantum mechanics can be characterized as a tensor product of C^* -algebras, providing a powerful tool to study quantum systems. This will also enhance the study of elementary particle theory and modelling of algebras of physical data in quantum mechanics. Also the results of this study can be essential building blocks for constructing more complex operators that describe physical systems and their evolution over time.

Category theory

These findings can enhance the study of category theory, where they are generalized to various categorical structures such as monoidal categories. This abstract perspective allows for a unified treatment of tensor product across different mathematical contexts.

Engineering and physics

This research results in tensor products can find extensive applications in engineering disciplines such as mechanical, fluid dynamics, and electromagnetism, where it can be used to model physical quantities with multiple components and their interactions. Also the results of this study can be applied in signal analysis and synthesis of signals and systems, particularly in the context of time-frequency analysis and wavelet theory.

Non-commutative geometry

The study in C*-algebra is intimately connected to non-commutative geometry, a branch of mathematics that generalizes geometric concepts to spaces where commutativity is not assumed. Thus results of this study can enhance the study of non-commutative aspects of space-time.

1.5 Definitions and Terminologies

In this section, the highlight of definitions of necessary terminologies that were used or relevant in this study are given.

Definition 1.5.1. Linear operator. (Bezandryns and Diagana, 2011).

Let X and Y be two normed spaces. A mapping $T : X \rightarrow Y$ is called a *linear operator* if:

$$(i) \quad T(x + y) = T(x) + T(y), \forall x, y \in X.$$

$$(ii) \quad T(\alpha x + \beta y) = \alpha T(x) + \beta T(y), \forall x, y \in X, \alpha, \beta \in \mathbb{K} \text{ where } \mathbb{K} \text{ is a scalar field.}$$

Definition 1.5.2. Bounded linear operator. (Bezandryns and Diagana, 2011).

A linear operator $T : X \rightarrow Y$ is said to be *bounded* if there exist a constant $c > 0$ such that;

$$\|Tx\| \leq c \|x\| \quad \forall x \in X.$$

This study considered algebras whose elements of operators on a Banach space are called *Operator algebras*. In this case, multiplication is the composition of operators.

Thus if X is a complex Banach space, $B(X)$ the Banach space of all bounded linear operators on X with respect to the operator norm in Banach algebra with the multiplication defined as composition of operators and the identity operator I being the unit element, then the operator norm of $T \in B(X)$ is given by;

$$\|T\| = \sup\{\|Tx\| : x \in X, \|x\| \leq 1\},$$

For $T, S \in B(X)$ we have that,

$$\|TSx\| \leq \|T\|\|S\|\|x\|$$

In which case,

$$\|TS\| \leq \|T\|\|S\|.$$

We also considered algebras of functions that is *Function algebras*. In this case, multiplication is pointwise.

Definition 1.5.3. Functional f . (Erdman, 2015).

A *functional f* , is a mapping that maps vectors to real value, that is, $f : L \rightarrow \mathbb{R}$. A linear function defined on a linear space L satisfies the following two properties:

(i) Additive: $f(x + y) = f(x) + f(y) \forall x, y \in L$.

(ii) Homogeneous: $f(\alpha x) = \alpha f(x)$.

The linear functional $f : A \rightarrow \mathbb{C} \forall A \in L$ is positive since $f(a^*a) \geq 0 \forall a \in A$ and is called a *positive linear functional*.

Definition 1.5.4. Normed space. (Halmos, 1982).

A *normed space* is a vector space B (assumed to be over the complex field \mathbb{C}) provided with a norm $\|\cdot\|$ satisfying

(i) $\|f\| \geq 0$.

(ii) $\|f\| = 0$ if and only if $f = 0$.

(iii) $\|\alpha f\| = |\alpha|\|f\|$.

(iv) $\|f + g\| \leq \|f\| + \|g\|$ for all $\alpha \in \mathbb{C}$ and $f, g \in B$.

Definition 1.5.5. Inner product. (Martin, 2009).

Let X be a vector space. An *inner product* is the function $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{R}$ which satisfies the following axioms, $\forall x, y \& z \in X$

- (i) $\langle x, x \rangle \geq 0$ with $\langle x, x \rangle = 0$ iff $x = 0$.
- (ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
- (iii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

The inner product can also be defined in terms of the norm, (Vasudeva and Shirali, 2017), as shown below;

$$\langle x, x \rangle = \|x\|^2.$$

Definition 1.5.6. Hilbert space. (Richard, 2019).

A (complex) Hilbert space H is a vector space on \mathbb{C} with a positive scalar product (or inner product) which is complete for the associated norm and which admits a countable orthonormal basis. The scalar product is denoted by $\langle \cdot, \cdot \rangle$ and the corresponding norm by $\|\cdot\|$.

In particular, note that for any $f, g, h \in H$ and $\alpha \in \mathbb{C}$ the following properties hold:

- (i) $\langle f, g \rangle = \overline{\langle g, f \rangle}$
- (ii) $\langle f, g + \alpha h \rangle = \langle f, g \rangle + \alpha \langle f, h \rangle$
- (iii) $\|f\|^2 = \langle f, f \rangle \geq 0$ and $\|f\| = 0$ if and only if $f = 0$.

Definition 1.5.7. Algebra. (Halmos and Spaces, 1974).

An algebra ϑ over a field \mathbb{K} is a space ϑ over \mathbb{K} such that for each ordered pair of elements $x, y \in \vartheta$ a unique product $xy \in \vartheta$ is defined with the following properties, $\forall x, y \in \vartheta$ and a scalar $\alpha \in \mathbb{N}$

- (i) $(xy)z = x(yz)$, (associativity).
- (ii) $x(y + z) = xy + xz, (x + y)z = xz + yz$, (distributivity).
- (iii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$ (associativity under scalar multiplication).

A normed algebra ϱ is a normed space which is an algebra such that $\forall x, y \in \varrho$ we have ;

$$\|xy\| \leq \|x\|\|y\|$$

A *Banach algebra* is a normed algebra that is complete, that is every Cauchy sequence in ϱ is convergent.

Definition 1.5.8. Involution. (Halmos, 1982).

If ϑ is an algebra, a mapping $*$: $\vartheta \rightarrow \vartheta$ defined by $x \rightarrow x^*$ is called an *involution* on algebra ϑ if it satisfies the following four conditions; $\forall x, y \in \vartheta, \lambda \in \mathbb{N}$

(i) $(x + y)^* = x^* + y^*$

(ii) $(\lambda x)^* = \bar{\lambda}x^*$

(iii) $(xy)^* = y^*x^*$

(iv) $(x^*)^* = x^{**} = x$

An algebra ϑ with an involution is called a ** - algebra*. If ϑ is a Banach algebra with an involution and $\forall x \in \vartheta, \|x^*x\| = \|x\|^2$, then ϑ is called *C* - algebra*.

Examples of *C**-algebras

(i) Let H be a complex Hilbert space. The collection of bounded linear operators on H , denoted by $B(H)$ is a *C** -algebra.

(ii) If n is any positive integer, we let $M_n(\mathbb{C})$ denote the set of $n \times n$ complex matrices. It is a *C**-algebra using the usual algebraic operations for matrices.

Definition 1.5.9. Elementary operator. (Okelo and Agure, 2011).

Let ϑ be a *C** - algebra, a mapping, T : $\vartheta \rightarrow \vartheta$ defined by $T(X) = \sum_{i=1}^n A_i X B_i$ for all $X \in \vartheta$ and for fixed A_i, B_i in $M(\vartheta)$, where $M(\vartheta)$ is the multiplier algebra is called an elementary operator.

Let $\vartheta = B(H)$ where H is a complete Hilbert space. Then the following are examples of elementary operators;

(i) The left multiplication operator $L_A : B(H) \rightarrow B(H)$ defined by $L_A(X) = AX$, $\forall X \in B(H)$ and for a fixed $A \in B(H)$

(ii) The right multiplication operator $R_B : B(H) \rightarrow B(H)$ defined by $R_B(X) = XB$, $\forall X \in B(H)$ and for a fixed $B \in B(H)$

- (iii) The general derivation (implemented by A,B) defined by $\delta_{A,B} = L_A - R_B$, for fixed $A, B \in B(H)$
- (iv) The basic elementary operator (implemented by A,B) defined by $M_{(A,B)} = AXB$, $\forall X \in B(H)$ and for fixed $A, B \in B(H)$
- (v) The Jordan elementary operator (implemented by A,B) defined by $U_{A,B} = AXB + BXA$, $\forall X \in B(H)$ and for fixed $A, B \in B(H)$
- (vi) The inner derivation $\delta_A = \delta_{AA}$, for a fixed $A \in B(H)$, where δ is general derivation.

Definition 1.5.10. Tensor product. (Halmos, 1982).

Let H and K be Hilbert spaces, a *tensor product* of H and K is a Hilbert space ℓ , together with a bilinear mapping $\varphi : H \otimes K \rightarrow \ell$ such that;

- (i) The set of all vectors $\{\varphi(x, y) : x \in H, y \in K\}$ form a total subset of ℓ , that is it's closed linear span is equal to ℓ and
- (ii) $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle \forall x_1, x_2 \in H$ and $\forall y_1, y_2 \in K$.

We refer (ℓ, φ) as Tensor product.

Definition 1.5.11. Numerical range. (Martin, 2009).

For an operator $T \in B(H)$. The *numerical range* of T is the set

$$W(T) = \{\lambda \in \mathbb{C} : \lambda = \langle Tx, x \rangle, \|x\| = 1, x \in H\}.$$

For an operator T on Hilbert space H , the following propositions holds (Blecher, 1988);

- (i) $W(T)$ is invariant under unitary similarity.
- (ii) $W(T)$ lies in a closed disc of radius $\|A\|$ centred at the origin.
- (iii) $W(T)$ contains all eigen values of T .
- (iv) $W(T^*) = \{\bar{\lambda} : \lambda \in W(T)\}$.
- (v) $W(I) = 1$.
- (vi) If $\alpha, \beta \in \mathbb{C}$ and T is bounded operator on H , then $W(\alpha T + \beta I) = \alpha W(T) + \beta$.

(vii) If H is of finite dimension then $W(T)$ is compact.

The *numerical radius* of T is represented as

$$w(T) = \sup\{|z| : z \in W(T)\}.$$

Definition 1.5.12. Types of numerical range. (Halmos, 1982).

The following are *types of numerical ranges*:

(i) The *maximal numerical range* of T is defined to be the set

$$W_0(T) = \{\lambda : \langle Tx_n, x_n \rangle \rightarrow \lambda, \text{ where } \|x_n\| = 1 \text{ and } \|Tx_n\| \rightarrow \|T\|\}.$$

(ii) The *algebraic numerical range* of T is the set;

$$W_{Alg}(T) = \{f(T) : f \text{ is linear functional on } \mathbb{C}^{n \times n}, f(I) = 1 = N^*(f)\}.$$

(iii) The *spatial numerical range*

Let V be a norm on Banach space Y and V^* be the dual norm defined by;

$$V^* = \max\{|Y^*X| : V(Y) \leq 1\}.$$

The *spatial numerical range* of T (also known as the *Bauer fields of values* of T) is defined by;

$$W_{spec}(T) = \{Y^*TX : V(X) = V^*(Y) = 1 = Y^*X\}.$$

Definition 1.5.13. Tracial geometrical mean. (Davies, 2007).

For two positive semidefinite matrices X and Y we define the *tracial geometrical mean* of X and Y by;

$$tgm(X, Y) = \sqrt{\sqrt{XY}\sqrt{X}}.$$

Definition 1.5.14. States (Halmos, 1982).

Let Ω be the complex unital Banach algebra with identity I and let $A \in \Omega$. We define the set of all *states* on Ω by

$$S(A) = \{f \in \Omega : f(I) = \|f\| = 1\}.$$

Definition 1.5.15. Finite rank operator. (Erdman, 2015).

let $u : H \rightarrow \mathbb{R}$ be a functional for a complex Hilbert space H , with dual H^* . A finite rank operator $u \otimes x : H \rightarrow H$ is defined by;

$$(u \otimes x)y = u(y)x, \forall y \in H,$$

where $u \in H^*$ and $x \in H$ is a unit vector, with;

$$\begin{aligned} \|u \otimes x\| &= \sup\{\|(u \otimes x)y\| : y \in H, \|y\| = 1\} \\ &= \sup\{\|u(y)x\| : y \in H, \|y\| = 1\} \\ &= \sup\{|u(y)| \|x\| : y \in H, \|y\| = 1\} \\ &= |u(y)|, \forall y \in H \end{aligned}$$

Definition 1.5.16. Rank one operator. (Stone, 1932).

Consider unit vectors $y, z \in H$. The *rank one operator*, $y \otimes z \in B(H)$ is defined as;

$$(y \otimes z)x = \langle x, z \rangle y \text{ for all unit vectors } x \in H.$$

Definition 1.5.17. Spectrum. (Davies, 2007).

Let H be a complex Hilbert space with inner product and T a bounded linear operator on H . The set

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$$

is referred to as the *spectrum* of T . The set

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$$

is called the *spectral radius* of the operator T .

The complement of $\sigma(T)$ is called the *resolvent set* of T denoted by

$$\rho(T) = \mathbb{C} - \sigma(T).$$

Definition 1.5.18. A tensor product subspace. (Okelo et al., 2010))

The tensor product $X \otimes Y$ is the subspace of $(B \times Y)'$ spanned by all elements of the form $x_i \otimes y_i$, where $x_i \in X$ and $y_i \in Y, i = 1, 2, \dots, n$. Thus, the typical form of $X \otimes Y$ is in the form of: $v = \sum_{i=1}^n \lambda_i x_i \otimes y_i$ where $n \in \mathbb{N}, \lambda_i \in \mathbb{K}, x_i \in Y, \forall i = 1, 2, \dots, n$ and \mathbb{K} is a scalar field. The mapping $(x, y) \in X \times Y \rightarrow x \otimes y \in X \otimes Y$ can be considered as a kind of 'multiplication', thus we have the following properties:

- (i) $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y.$
- (ii) $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2.$
- (iii) $\lambda(x \otimes y) = (\lambda x) \otimes y = x \otimes (\lambda y).$
- (iv) $0 \otimes y = y \otimes 0 = 0.$

Definition 1.5.19. A tensor norm. (Mathieu, 2001)

A tensor norm on $X \otimes Y$ is a norm $\|\cdot\|$ on $X \otimes Y$ such that $\|X \otimes Y\| \leq \|X\|\|Y\|$. The consequence noticed on the condition is that from the triangle inequality $\forall v \in X \otimes Y$ with $v = \sum_{i=1}^n x_i \otimes y_i$ being a representation of v , then;

$$\|v\| = \|\sum_{i=1}^n x_i \otimes y_i\| \leq \sum_{i=1}^n \|x_i\|\|y_i\|.$$

Then it follows that

$$\|v\| \leq \inf\{\sum_{i=1}^n \|x_i\|\|y_i\|\}$$

such that $v = \sum_{i=1}^n x_i \otimes y_i$ since it represents every v .

Definition 1.5.20. Maximal and minimal tensor products (Bruce, 1977)

Let A and B be two C*-algebras, and denote by $A \odot B$ their algebraic tensor product. A C*-norm on $A \odot B$ is a norm of involutive algebra such that $\|x^*x\| = \|x\|^2$ for $x \in A \odot B$. There are two natural ways to define C*-norms on the *-algebra $A \odot B$. Recall first that a representation of a C*-algebra in a Hilbert space H is a homomorphism from A into the C*-algebra $B(H)$ of all bounded operators on H . An important fact to keep in mind is that homomorphism of C*-algebras are automatically contractive maps. The maximal C*-norm is defined by;

$$\|x\|_{Max} = Sup\{\|\pi(x)\|\}$$

where π runs over all homomorphism from $A \odot B$ into some $B(H)$. The maximal tensor product is the completion $A \otimes_{Max} B$ of $A \odot B$ for this C*-norm.

The minimal tensor product is defined by taking specific homomorphisms, namely of the form $\pi_1 \otimes \pi_2$ where π_1 is a representation of A in some Hilbert space H and π_2 is a representation of B in some KH . Thus we define;

$$\|x\|_{min} = \sup\|(\pi_1 \otimes \pi_2)(x)\|$$

where π_1, π_2 run over all representations of A and B respectively. The minimal tensor product is the completion $A \otimes_{min} B$ of $A \odot B$ for this C^* -norm. It is shown that when π_1 and π_2 are faithful, we have;

$$\|x\|_{min} = \|(\pi_1 \otimes \pi_2)(x)\|.$$

Therefore, if A and B are purely represented as C^* -subalgebras of $B(H)$ and $B(K)$ respectively, then $A \otimes_{min} B$ is the closure of $A \odot B$, viewed as a subalgebra of $B(H \otimes K)$. It is the reason why $\|\cdot\|_{min}$ is also called the spatial tensor product.

If B and C are C^* -algebras, the algebraic tensor product $B \odot C$ can be completed with respect to the least C^* -cross norm to give the C^* -tensor product $B \otimes C$. This is the only tensor product that this thesis will consider.

Definition 1.5.21. Perturbation theory for self-adjoint operators. (Richard, 2019)

Let $(A, D(A))$ be a self-adjoint operator in a Hilbert space H . The invariance of the self-adjoint property under the addition of a linear operator B is an important issue, especially in relation with quantum mechanics and this study. First of all, observe that if $(A, D(A))$ and $(B, D(B))$ are symmetric operators in H and if $D(A) \cap D(B)$ is dense in H , then $A + B$ defined on this intersection is still a symmetric operator. Indeed, one has for any $f, g \in D(A) \cap D(B)$

$$\langle (A + B)f, g \rangle = \langle Af, g \rangle + \langle Bf, g \rangle = \langle f, Ag \rangle + \langle f, Bg \rangle = \langle f, (A + B)g \rangle$$

Definition 1.5.22. Norm on a C^* -algebra. (Galles and Arratum, 2021)

A norm on a $*$ -algebra A is called a C^* -norm if it satisfies;

$$\|x\| = \|x^*\|, \|xy\| \leq \|x\|\|y\| \text{ and } \|xx^*\| = \|x\|^2 \text{ for any } x, y \in A.$$

The completion of A of $(A, \|\cdot\|)$ then becomes a C^* -algebra. Let A_1, A_2 be two C^* -algebras. Their algebraic tensor $A_1 \otimes A_2$ is a $*$ -algebra for the natural operations defined by;

$$(a_1 \otimes a_2).(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2 \text{ and } (a_1 \otimes a_2)^* = a_1^* \otimes a_2^*.$$

Thus a norm $\|\cdot\|$ on $A_1 \otimes A_2$ is a norm if it satisfies;

$$\|x\| = \|x^*\|, \|xy\| \leq \|x\|\|y\| \text{ and } \|xx^*\| = \|x\|^2 \text{ for any } x, y \in A_1 \otimes A_2.$$

It follows that any C*-norm $\|\cdot\|$ on $\|A_1 \otimes A_2\|$ automatically satisfies;

$$\|a_1 \otimes a_2\| = \|a_1\|\|a_2\| \forall a_1 \in A_1 \forall a_2 \in A_2$$

The following are properties of C*-norm according to Pisier (2020)

- (i) There is a minimal C*-norm $\|\cdot\|_{min}$ and maximal one $\|\cdot\|_{max}$, so that any C*-norm $\|\cdot\|$ on $A_1 \otimes A_2$ must satisfy

$$\|x\|_{min} \leq \|x\| \leq \|x\|_{max} \forall x \in A_1 \otimes A_2$$

We denote by $A_1 \otimes A_2$ (resp $A_1 \otimes_{max} A_2$) the completion of $A_1 \otimes A_2$ for norm $\|\cdot\|_{min}$ (resp $\|\cdot\|_{max}$)

- (ii) The maximal C*-norm is easy to describe. We simply write

$$\|x\|_{max} = \sup \|\pi(x)\|_{B(H)}$$

Where the supremum of all possible Hilbert space H and all possible *-homomorphism $\pi : A_1 \otimes A_2 \rightarrow B(H)$

- (iii) A C*-algebra A is called nuclear if for any C*-algebra B we have $\|\cdot\|_{min} = \|\cdot\|_{max}$ on $A \otimes B$ that is,

$$A \otimes_{min} B = A \otimes_{max} B$$

CHAPTER TWO

LITERATURE REVIEW

2.1 Basic Elementary Operator

Different researchers have gained interest in determining the norm of the basic elementary operator in different spaces due to its wide application and use in different areas of physics and applied mathematics.

Timoney (2001) determined the general norm of basic elementary operator in C^* -algebra and obtained theorem 2.1.1;

Theorem 2.1.1:(Timoney, 2001)

Let \mathcal{A} be C^* -algebra. Then;

$$\|M_{A,B}\| = \sup\{\|M_{A,B}(X)\| : X \in B(H)\} = \sup\{\|AXB\| : X \in B(H)\}$$

where X denotes the set of unitaries in $B(H)$.

The above work of Timoney (2001) established the foundation for determining the upper bounds of the norm of the Jordan elementary operator and finite length elementary operator in the tensor product of C^* -algebras which were among the main objectives in this study.

Boumazgour (2008) obtained the norm inequality for the sum of two basic elementary operators and achieved theorem 2.1.2;

Theorem 2.1.2:(Boumazgour, 2008)

If A, B, C and D are operators in $B(H)$, then;

$$\|M_{A,B} + M_{C,D}\| \leq [(\max\{\|B\|^2, \|D\|^2\} + \|BD^*\|)(\max\{\|A\|^2, \|C\|^2\} + \|C^*A\|)]^{\frac{1}{2}}.$$

From Boumazgour (2008) it's clear that only the upper bound of the norm of the sum of two basic elementary operators in C^* -algebra has been determined and this leaves a gap for the calculation of the lower bound so as to obtain the equality condition.

Further, Okelo (2011) using numerical range and finite rank operators proved lemma 2.1.3 in basic elementary operator;

Lemma 2.1.3 : (Okelo, 2011)

Let H be a Hilbert space, $B(H)$ the algebra of bounded linear operators on H and a norm-attainable basic elementary operator $M_{A,B} : B(H) \rightarrow B(H)$ be defined by $M_{A,B} = AXB$, $\forall X \in B(H)$ (where A and B are norm-attainable operators on $B(H)$). Then;

$$\|M_{A,B}\| = \|A\|\|B\|.$$

From the above lemma 2.1.3, the equality condition have been determined but it gives room to extending the norm of the basic elementary operator in tensor product of C^* -algebras and also extending the work of Boumazgour (2008) on the norm of the of two basic elementary operators in tensor product of C^* -algebras

Also Okelo (2011) obtained the norm of the sum of two tensor products of bounded linear operators in a Hilbert space and arrived at the following theorem 2.1.4;

Theorem 2.1.4 : (Okelo, 2011)

If $A, B \in B(H)$ and $A \otimes B$ denote the tensor product of A and B then;

$$\|A \otimes B + B \otimes A\| \leq \sqrt{2\|A\|^2\|B\|^2 + 2\|B^*A\|}.$$

This work of Okelo (2011) give this study ideal property that was applied in the determination of the norms in different types of elementary operators in tensor product of C^* -algebras.

Okelo and Agure (2011) also used the finite rank operators to determine the norm of the basic elementary operator. Okelo and Agure (2011) arrived at Lemma 2.1.5, whose proof also created the basis for the research more so in the application of rank one operators.

Lemma 2.1.5: (Okelo and Agure, 2011)

Let H be a Hilbert space, $B(H)$ the algebra of bounded linear operators on H . If $M_{T,S} : B(H) \rightarrow B(H)$ is defined by $M_{T,S}(X) = TXS$, $\forall X \in B(H)$ where A and B are fixed in $B(H)$ then;

$$\|M_{T,S}\| = \|T\|\|S\|, \text{ for } \|X\| = 1 \text{ where } X(x) = x, \forall \text{ unit vectors } x \in H.$$

Bachir et al. (2012) estimated the norm of the sum of two basic elementary operator $M_{A_1,B_1} + M_{A_2,B_2}$, where A_1, A_2, B_1, B_2 are bounded linear operators on a normed space E and M_{A_1,B_1} is the basic elementary operator defined on $B(E)$ by $M_{A_1,B_1}(X) = A_1XB_1$. Bachir et al. (2012) also gave necessary and sufficient conditions on the operators A_1, A_2, B_1, B_2 under

which $M_{A_1, B_1} + M_{A_2, B_2}$ attains its optimal value $\|A_1\|\|B_1\| + \|A_2\|\|B_2\|$ and they first determined the generalized numerical range of A relative to B denoted by $\mathfrak{M}(A)_B$;

For $A, B \in B(E)$, let $S_E(B) = \{(x_n)_n : x_n \in S_E, \|Bx_n\| \rightarrow \|B\|\}$, then the set

$$\mathfrak{M}(A)_B = \{\lim \varphi_n(Ax_n) : (x_n)_n \in S_E(B), \varphi_n \in D(Bx_n)\},$$

where S_E is the unit sphere of E and E^* the dual topological space of E and D is the normalized duality mapping from E to E^* given by

$$D(X) = \{\varphi \in E^* : \varphi(X) = \|x\|^2, \|\varphi\| = \|x\|\} \forall x \in E$$

It is known that $\mathfrak{M}(A)_B$ is a non empty closed subset of \mathbb{K} and $M(A)_B \in \mathfrak{M}(A)_B \in \overline{W(A)_B}$, where $M(A)_B$ is the maximal spatial numerical range of $A \in B(E)$ relative to B and $\overline{W(A)_B}$ is the closure of the spatial numerical range $W(A)_B$ of $A \in B(E)$ relative to B .

The definition of $\mathfrak{M}(A)_B$ can be written, with respect to the semi-inner product $[\cdot, \cdot]$ as

$$\mathfrak{M}(A)_B = \{\lim [Ax_n, Bx_n] : (x_n)_n \in S_E(B)\}$$

with respect to an inner product $\langle \cdot, \cdot \rangle$ as

$$\mathfrak{M}(A)_B = \{\lim \langle Ax_n, Bx_n \rangle : (x_n)_n \in S_E(B)\}.$$

Using the concept of generalized maximal numerical range Bachir et al. (2012) arrived at theorem 2.1.3 below

Theorem 2.1.6: (Bachir *et al.*, 2012)

Let A_1, A_2, B_1, B_2 be operators in $B(E)$. If $\|A_1\|\|A_2\| \in \mathfrak{M}(A_1)_{A_2} \cup \mathfrak{M}(A_2)_{A_1}$ and $\|B_1\|\|B_2\| \in \mathfrak{M}(B_1)_{B_2} \cup \mathfrak{M}(B_2)_{B_1}$, then

$$\|M_{A_1, B_1} + M_{A_2, B_2}\| = \|A_1\|\|B_1\| + \|A_2\|\|B_2\|$$

As a consequence of the above theorem Bachir *et al.* (2012) found that;

Corollary 2.1.7: (Bachir *et al.*, 2012)

Let E be a normal space and $A, B \in B(E)$. Then , the following assertions hold:

(i) If $\|A\|\|B\| \in \mathfrak{M}(A)_B$, then $\|A + B\| = \|A\|\|B\|$;

(ii) if $\|A\| \in \mathfrak{M}(I)_A$ and $\|B\| \in \mathfrak{M}(I)_B$, then $\|M_{A, B} + I\| = 1 + \|A\|\|B\|$

Finally, Wickstead (2015) showed that if E is atomic Banach lattice with an order continuous norm, $A, B \in \mathcal{L}^r(E)$ and $M_{A,B}$ is an operator on $\mathcal{L}^r(E)$ defined by $M_{A,B}(T) = ATB$, then $\|M_{A,B}\|_r = \|A\|_r \|B\|_r$ thus he proved theorem 2.1.8 below;

Theorem 2.1.8: (Wickstead, 2015)

If E is an atomic Banach lattice with an order continuous norm and $A, B \in \mathcal{L}^r(E)$, then;

$$|M_{A,B}| = M_{|A|,|B|}$$

As a consequence Wickstead (2015) also found that;

Corollary 2.1.9: (Wickstead, 2015)

If E is an atomic Banach lattice with an order continuous norm and $A, B \in \mathcal{L}^r(E)$, then;

$$\|M_{A,B}\|_r = \|A\|_r \|B\|_r$$

The basic elementary operator literature mentioned above provided the necessary context for methodology, results, proper assertions, and the identification of the right solutions of the study.

2.2 Elementary Operator of Finite Length

This section considers the elementary operator of length $n = 2$ to the finite case that is for any $n \in \mathbb{N}$.

Mathieu (2001) determined the norm of elementary operator T on Calkin algebra. Mathieu (2001) proved the theorem 2.2.1 below;

Theorem 2.2.1: (Matheiu , 2001)

Let T be an elementary operator on $B(H)$. Then;

$$\|T\| = \inf \left\{ \left\| \sum_{i=1}^n A_i A_i^* \right\|^{\frac{1}{2}} \left\| \sum_{i=1}^n B_i^* B_i \right\|^{\frac{1}{2}} \right\},$$

where the infimum is taken over all representations of T where $T = \sum_{i=1}^n M_{A_i, B_i}$.

Timoney (2007) also determined the norm of an elementary operator on a C^* -algebra using the notion of matrix valued numerical ranges and a kind of geometrical mean for a positive matrices, tracial geometric mean. Timoney (2007) arrived at theorem 2.2.2;

Theorem 2.2.2: (Timoney, 2007)

For $A = (A_1 \dots A_n) \in B(H)^n$ (a row matrix of operators $A_i \in B(H)$),

$B = (B_1 \dots B_n) \in B(H)^n$ (a matrix of operators $B_i \in B(H)$) and

$T_{A,B}(X) = \sum_{i=1}^n A_i X B_i \forall X \in B(H)$ elementary operator, we have;

$$\|T\| = \sup\{tgmQ(A^*, \varepsilon), Q(B, \eta)\}; \varepsilon, \eta \in B(H)\} \text{ and } \|\varepsilon\| = \|\eta\| = 1.$$

From above theorems 2.2.1 and 2.2.2 of Mathieu (2001) and Timoney (2007), it is clear this study extended the determination of the norm of finite length elementary operator in tensor product of C^* -algebras.

On their part, Hong-ke *et al.* (2008) determined the norm of elementary operator of finite length and arrived to the following theorem 2.2.3;

Theorem 2.2.3:(Hong-ke *et al.*, 2008)

Let A_i and B_i for $1 \leq i \leq n$, be bounded linear operators acting on a separable Hilbert space H . Then;

$$\begin{aligned} \sup\{\|\sum_{i=1}^n A_i X B_i\|, \forall X \in B(H), \|X\| = 1\} &= \sup\{\|\sum_{i=1}^n A_i U B_i\|, U U^* \\ &= U^* U = I U \in B(H)\}. \end{aligned}$$

Moreover Hong-ke *et al.* (2008) proved that there exists an operator X_0 with $\|X_0\| = 1$ such that;

$$\|\sum_{i=1}^n A_i X_0 B_i\| = \sup\{\|\sum_{i=1}^n A_i X B_i\|, \forall X \in B(H), \|X\| \leq 1\}$$

if and only if there exists a unitary $U_0 \in B(H)$ such that

$$\|\sum_{i=1}^n A_i U_0 B_i\| = \|\sum_{i=1}^n A_i X B_i\|, \forall X \in B(H), \|X\| \leq 1.$$

Further, Nyamwala and Agure (2008) used the spectral resolution theorem to calculate the norm of elementary operator induced by normal operators in a finite dimensional Hilbert space. They gave the following theorem 2.2.4.

Theorem 2.2.4:(Nyamwala and Agure, 2008)

Let $T_{A,B} : B(H) \rightarrow B(H)$ be an elementary operator defined by;

$$T_{A,B} = \sum_{i=1}^n A_i X B_i, \forall X \in B(H)$$

and A_i, B_i be normal operators and H a finite m -dimensional Hilbert space. Then

$\|T\| = (\sum_{j=1}^k |\alpha_{ij}|^2 |\beta_{ij}|^2)^{\frac{1}{2}}$ where α_{ij}, β_{ij} are distinct eigen values of A_i and B_i respectively.

Okelo and Agure (2008) used the spectral resolution theorem to obtain the norm of elementary operator. They arrived at the following theorem 2.2.5;

Theorem 2.2.5: (Okelo and Agure, 2008)

Let $T : B(H) \rightarrow B(H)$ be an elementary operator defined by;

$$T(X) = \sum_{n=1}^k A_n X B_n, A_n, B_n \in B(H), X \in B(H)$$

where A_i, B_i are diagonal operators induced by α_{ni} and β_{ni} respectively and H a finite dimensional complex Hilbert space. Then T is bounded and

$$\|T\| = (\sum \sum_{n=1}^k |\alpha_{ni}|^2 |\beta_{ni}|^2).$$

King'ang'i *et al.* (2014) determined the norm of elementary operator for finite-dimensional separable Hilbert space $W \in B(H)$ with $\|W\| = 1$ and $W(x) = x$ for all unit vectors $x \in H$. They gave the following result in the theorem 2.2.6;

Theorem 2.2.6: (King'ang'i *et al.*, 2014)

Let H be a complex Hilbert space and $B(H)$ be algebra of all bounded linear operators on H . Let E_2 be the elementary operator on $B(H)$. If for an operator $W \in B(H)$ with $\|W\| = 1$, we have $W(x) = x$ for all unit vectors $x \in H$, then;

$$\|E_2\| = \sum_{i=1}^2 \|T_i\| \|S_i\|.$$

King'ang'i (2017) employed the concept of the maximal numerical range of T^*S relative to S to determine the lower bound of the norm of an elementary operator of length two. King'ang'i (2017) proved the following theorem 2.2.7;

Theorem 2.2.7: (King'ang'i, 2017)

Let E_2 be an elementary operator of length two on $B(H)$. Then;

$$\|E_2\| \geq \text{Sup}_{\lambda \in W_S(T^*S)} \| \|S_1\| T_1 + \frac{\lambda}{\|S_1\|} T_2 \|.$$

King'ang'i (2017) also determined the conditions under which the norm of an elementary operator of length two is expressible in terms of the norms of its coefficients operators by proving the following Corollary 2.2.8 and theorem 2.2.9;

Corollary 2.2.8: (King'ang'i, 2017)

Let H be a complex Hilbert space and T_i, S_i be bounded linear operators on H for $i = 1, 2$. Let $0 \in W_{S_1}(S_1^*S_2) \cup W_{S_2}(S_1^*S_2)$. Then;

$$\|E_2\| \geq \|T_1\| \|S_1\|,$$

where E_2 is the elementary operator of length two.

Theorem 2.2.9: (King'ang'i, 2017)

Let H be a complex Hilbert space and T_i, S_i be bounded linear operators on H for $i = 1, 2$. Let E_2 be an elementary operator of length two. If $\|T_1\| \|T_2\| \in W_{T_1}(T_2T_1^*)$ and $\|S_1\| \|S_2\| \in W_{S_2}(S_1^*S_2)$, then;

$$\|E_2\| = \sum_{i=1}^n \|T_i\| \|S_i\|.$$

Finally, King'ang'i (2018) presented more results on the norm of this operator by employing the concept of the Stampfli's maximal numerical range. He determined the lower bound of the norm of the operator E_2 and the conditions necessary to express the norm of E_2 in the form $\|E_2\| = \sum_{i=1}^2 \|T_i\| \|S_i\|$ thereby obtaining the following two results.

Theorem 2.2.10: (King'ang'i, 2018)

Let E_2 be an elementary operator on $B(H)$ and $S_1, S_2 \in B(H)$. If $\lambda_i \in W_0(S_i)$ for each $\lambda_i \in \mathbb{C}$, $i = 1, 2$ then we have;

$$\|E_2\| \geq \sup_{\lambda_i \in W_0(S_i)} \{ \|\sum_{i=1}^n \lambda_i T_i\| : T_i \in B(H), i = 1, 2 \}.$$

Theorem 2.2.11: (King'ang'i, 2018)

Let E_2 be an elementary operator on $B(H)$ and $S_1, S_2 \in B(H)$ for $i = 1, 2$. If $\|S_i\| \in W_0(S_i)$ and $\|T_i\| \in W_0(T_i)$ for $i = 1, 2$, then;

$$\|E_2\| = \sum_{i=1}^2 \|T_i\| \|S_i\|.$$

Also using the concept of finite rank operator Kawira *et al.* (2018) determined the norm of the elementary operator of finite length in C^* -algebra and obtained the following theorem 2.2.12;

Theorem 2.2.12: (Kawira *et al.*, 2018)

Let H be complex Hilbert spaces and $B(H)$ be the algebra of bounded linear operators on H . Let E_n be elementary operator on $B(H)$. If $\forall X \in B(H)$ with $\|X\| = 1$, we have $X(f) = f$ for all unit vector $f \in H$ then;

$$\|E_n\| = \sum_{i=1}^n \|A_i\| \|B_i\|, n \in \mathbb{N}.$$

The above literature on norm of finite length elementary operator in different spaces gave the study different spaces and methodologies that were used to determine the norm of the finite-length elementary operator in tensor products of C^* -algebras.

2.3 Jordan Elementary Operator

While working on a prime C^* -algebras Mathieu (1990) determined the lower bound of the norm of the Jordan elementary operator. He proved the following theorem 2.3.1;

Theorem 2.3.1: (Mathieu, 1990)

Let Λ be a prime C^* -algebra and A, B its elements. Then;

$$\|AXB + BXA\| \geq \frac{2}{3} \|A\| \|B\|.$$

Also Garcia and Palacios (1994) determined the norm of Jordan operator for JB^* -algebras as shown in theorem 2.3.2;

Theorem 2.3.2: (Garcia and Palacios , 1994)

Let Z be a JB^* -algebra and A, B its elements. Then;

$$\|AXB + BXA\| \geq \frac{1}{20412} \|A\| \|B\|.$$

Stacho and Zalar (1996) also determined the norm of Jordan elementary operator for standard operator algebras and proved the following theorem 2.3.3;

Theorem 2.3.3: (Stacho and Zalar, 1996)

Let Λ be standard operator algebra on a Hilbert space H . If $A, B \in \Lambda$, then the uniform estimate;

$$\|AXB + BXA\| \geq 2(\sqrt{2} - 1) \|A\| \|B\|.$$

Moreover, Bunce *et al.* (1997) both sharpened and extended the result of Garcia and Palacios., 1994 for any prime JB^* -triple. They arrived to the following result in form of the next proposition 2.3.4.

Proposition 2.3.4: (Bunce *et al.*, 1997)

Let Λ be a finite dimensional cartan factor. Then;

$$\|M_{AB}\| \geq \frac{1}{6}\|A\|\|B\|, \forall A, B \in \Lambda.$$

Working in different spaces that is prime C^* -algebra, JB^* -algebra, standard operator algebra, and prime JB^* -triple and also using different methods gave this research a gap. Where this study determined the norm of Jordan elementary operator in tensor product of C^* -algebras.

Stacho and Zalar (1998) also determined the norm of Jordan elementary operator for the algebra of symmetric operators acting on a Hilbert space. Stacho and Zalar (1998) proved result as shown in the theorem 2.3.5 that follows.

Theorem 2.3.5: (Stacho and Zalar, 1998)

Let $A, B \in \text{symm}(H)$. Then;

$$\|M_{A,B} + M_{B,A}\| \geq \|A\|\|B\|.$$

In this case Stacho and Zalar (1998) were interested in the quantum mechanical Jordan algebra $\text{symm}(H) = \{A \in B(H); A^* = A\}$

On their part, Barraa and Boumazgour (2001) used the concept of the maximal numerical range and finite rank operators in prime C^* -algebras to determine norm of Jordan elementary operator and managed to arrive at theorem 2.3.6 ;

Theorem 2.3.6: (Barraa and Boumazgour, 2001)

Let $T, S \in B(H)$ with $S \neq 0$ then;

$$\|U_{T,S}\| \geq \sup_{\lambda \in W_S(T^*S)} \left\{ \| \|S\|T + \frac{\bar{\lambda}}{\|S\|}S \| \right\},$$

where $W_S(T^*S) = \{ \lambda \in \mathbb{C} : \exists x_n \in H, \|x_n\| = 1, \lim_{n \rightarrow \infty} \langle T^*Sx_n, x_n \rangle = \lambda, \lim_{n \rightarrow \infty} \|Sx_n\| = \|S\| \}$.

As a consequence of this, Barraa and Boumazgour(2001) proved the following Corollary 2.3.7.

Corollary 2.3.7: (Barraa and Boumazgour, 2001)

Let H be a complex Hilbert space and T, S be bounded linear operators on H . If $0 \in W_S(T^*S) \cup W_T(S^*T)$, then;

$$\|U_{T,S}\| \geq \|T\|\|S\|.$$

Barraa and Boumazgour(2001) also proved the following proposition 2.3.8.

Proposition 2.3.8:(Barraa and Boumazgour, 2001)

Let H be a complex Hilbert space and T, S be bounded linear operators on H . If $\|T\|\|S\| \in W_S(T^*S) \cap W_T(S^*T)$, then;

$$\|U_{T,S}\| = 2\|T\|\|S\|.$$

Timoney (2003) determined the norm of an elementary operator on a C^* -algebra using the notion of matrix-valued numerical ranges and a kind of geometrical mean for positive matrices and tracial geometric mean.

Theorem 2.3.9: (Timoney, 2003)

Assume that H is a two dimensional Hilbert space and $A, B \in B(H)$. Let $U_{A,B} = M_{A,B} + M_{B,A}$. Then;

$$\|U_{A,B}\| \geq \|A\|\|B\|.$$

As a consequence Timoney (2003) also shown that;

Theorem 2.3.10: (Timoney, 2003)

Let $A = B(H)$ and let T be an elementary operator on $B(H)$. Then we have equality;

$$\|T\| \leq \|T\|_{cb} \leq \frac{1}{2}(\|\sum_{j=1}^n A_j A_j^*\| + \|\sum_{j=1}^n B_j B_j^*\|)$$

if and only if the intersection $W_{me}(a_1^*, a_2^*, \dots, a_n^*) \cap W_{me}(b_1, b_2, \dots, b_n)$ is empty.

This work of Timoney (2003) gave this work the bases of calculating the lower bound of the norm of Jordan elementary operator in tensor product of C^* -algebras

On their part, Blanco *et al.* (2004) determined norm of a Jordan elementary operator on a C^* -algebra using the notion of matrix valued numerical ranges and a kind of geometrical mean for positive matrices and tracial geometric mean.

Theorem 2.3.11: (Blanco *et al.*, 2004)

Let H be a two dimensional complex Hilbert space, $B(H)$ the algebra of bounded linear operators on H . Let $M_{A,B} : B(H) \rightarrow B(H)$ be defined by

$M_{A,B}(X) = AXB + BXA, \forall X \in B(H)$ where A and B are fixed in $B(H)$ and e_1, e_2 are orthonormal basis of H then;

$$\|AXB + BXA\| = \|A\|\|B\|.$$

On his part Saddik(2004) expounded on the conditions that must be satisfied for $\|AXB + BXA\| \geq 2(\sqrt{2}-1)\|A\|\|B\|$ in Stacho and Zalar (1996) to hold by using algebraic numerical range and Joint spatial numerical range and obtained theorem 2.3.12

Theorem 2.3.12: (Saddik, 2004)

Let E be a Hilbert space, then the lower bound of Jordan Elementary Operator

$$\|U_{A,B} \setminus J\| \geq 2(\sqrt{2} - 1)\|A\|\|B\|$$

holds if one of the following two conditions is satisfied.

- (i) J is a Standard Operator algebra of $B(E)$ and $A, B \in J$
- (ii) J is norm ideal of $B(E)$ and $A, B \in B(E)$.

Similarly, Xiaoli and Guoxing (2008) discussed the norm of $U_{A,B}$ by showing that if $\dim H = 2$ and $\|U_{A,B}\| = \|A\|\|B\|$, then either AB^* or B^*A is 0 and also gave some examples of Jordan elementary operators $U_{A,B}$ such that

$$\|U_{A,B}\| = \|A\|\|B\|, \text{ but } AB^* \neq 0 \text{ } B^*A \neq 0,$$

which answer negatively a question posed by Boumazgour (2008). Generally, Xiaoli and Guoxing (2008) answered the question of the conditions under which the norm of Jordan elementary operator in C^* -algebra expressible in terms of the norm of its coefficient operators.

Further, Nyamwala (2009) calculated the norm of symmetric two sided multiplication operator $T_{PQ_k}E = PEQ_k + Q_kEP$ defined on C^* -algebra $C^*(P, Q_k, 1)$ generated by P and Q_k .

Theorem 2.3.13: (Nyamwala, 2009)

Let $T_{PQ_k}E = PEQ_k + Q_kEP$ be a symmetrized two sided multiplication operator then;

$$\lim_{A_k \rightarrow 0} \|T_{PQ_k}\| = 2 \text{ and } \lim_{A_k \rightarrow 0} \|T_{PQ_k}\|_2 = 2.$$

Finally, Okelo *et al.* (2010) determined the norm of Jordan elementary operator

$M_{A,B}(X) = AXB + BXA, \forall X \in B(H)$ and A, B fixed in $B(H)$. In particular, Okelo *et al* (2010) proved the theorem 2.3.14;

Theorem 2.3.14: (Okelo *et al.*, 2010)

Consider a norm-attainable Jordan elementary operator $M_{A,B} : B(H) \rightarrow B(H)$ defined by $M_{A,B}(X) = AXB + BXA, \forall X \in B(H)$. Assume $A, B \in B(H)$ are norm-attainable such that $A = \phi Q$ and $B = \phi R$ where $Q = |A|$ and $R = |B|$ and ϕ is unitary in $B(H)$. Then;

$$\|U_{A,B}\| \geq \|A\| \|B\|.$$

The literature from Nyamwala (2009) and Okelo *et al.* (2010) also highlighted the norm of Jordan elementary operator in different spaces using different methods but there exists no literature norm of Jordan elementary operator in the tensor product of C^* -algebras. This created the need for determining the norm of Jordan elementary operator in tensor product of C^* -algebras due to its wide range of use in physics. Thus this was one of the study's objective that was solved.

2.4 Elementary Operator in Tensor Product

In 2018, Muiruri *et al.* defined the elementary operator in a tensor product of C^* -algebras and obtained the following;

Let $H \otimes K$ be tensor product of complex Hilbert spaces H and K and $B(H \otimes K)$, the set of bounded linear operators on $H \otimes K$, $A \otimes B, C \otimes D$ being fixed elements of $B(H \otimes K)$, where $A, C \in B(H)$, the set of bounded linear operators on H and $B, D \in B(K)$, the set of bounded linear operators on K . Then:-

An elementary operator, $T_n : B(H \otimes K) \rightarrow B(H \otimes K)$, is defined as;

$$T_n(X \otimes Y) = \sum_{i=1}^n A_i \otimes B_i(X \otimes Y)C_i \otimes D_i, \forall X \otimes Y \in B(H \otimes K),$$

$A_i \otimes B_i, C_i \otimes D_i$ being fixed elements of $B(H \otimes K)$, where $A_i, C_i \in B(H)$, the set of bounded linear operators on H and $B_i, D_i \in B(K)$, the set of bounded linear operators on K .

When $n = 1$ we obtain the basic elementary operator, $M_{A \otimes B, C \otimes D} : B(H \otimes K) \rightarrow B(H \otimes K)$, defined as;

$$M_{A \otimes B, C \otimes D}(X \otimes Y) = A \otimes B(X \otimes Y)C \otimes D, \forall X \otimes Y \in B(H \otimes K).$$

When $n = 2$ then we obtain an elementary operator of length two which is defined by;

$$T_2(X \otimes Y) = A_1 \otimes B_1(X \otimes Y)C_1 \otimes D_1 + A_2 \otimes B_2(X \otimes Y)C_2 \otimes D_2, \forall X \otimes Y \in B(H \otimes K)$$

and $A_i \otimes B_i, C_i \otimes D_i$ being fixed elements of $B(H \otimes K)$, where $A_i, C_i \in B(H)$, the set of bounded linear operators on H and $B_i, D_i \in B(K)$, the set of bounded linear operators on K .

The Jordan elementary operator, $U_{A \otimes B, C \otimes D} : B(H \otimes K) \rightarrow B(H \otimes K)$, is defined as;

$$U_{A \otimes B, C \otimes D}(X \otimes Y) = A \otimes B(X \otimes Y)C \otimes D + C \otimes D(X \otimes Y)A \otimes B, \\ \forall X \otimes Y \in B(H \otimes K),$$

$A \otimes B, C \otimes D$ being fixed elements of $B(H \otimes K)$, where $A, C \in B(H)$, the set of bounded linear operators on H and $B, D \in B(K)$, the set of bounded linear operators on K .

These definitions resulted in the determination of the norm of basic elementary operator in a tensor product using different methods.

Using the rank one operator and properties of tensor product Muiruri *et al.* (2018), proved the following theorem 2.4.1;

Theorem 2.4.1:(Muiruri *et al.*, 2018)

Let H and K be complex Hilbert spaces and $B(H \otimes K)$ be the set of bounded linear operators on $H \otimes K$. Then $\forall X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$, we have;

$$\|M_{A \otimes B, C \otimes D}\| = \|A\| \|B\| \|C\| \|D\|.$$

where A, C and B, D are fixed elements in $B(H)$ and $B(K)$ respectively.

Consequently, Muiruri *et al.* (2018) related the norm of basic elementary operator in tensor product and the usual norm in this elementary operator in C^* -algebra and arrived at the corollary 2.4.2;

Corollary 2.4.2:(Muiruri *et al.*, 2018)

Let H and K be complex Hilbert spaces and $B(H \otimes K)$ be the set of bounded linear operators on $H \otimes K$. Then $\forall X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$, we have;

$$\|M_{A \otimes B, C \otimes D}\| = \|M_{A, C}\| \|M_{B, D}\|.$$

where $M_{A, C}$ and $M_{B, D}$ are basic elementary operators in $B(H)$ and $B(K)$ respectively.

Daniel *et al.* (2022) extended the result of Muiruri *et al.* (2018) by using the stampfli's maximal numerical range to determine the norm of basic elementary operator in a tensor product and they obtained the following theorem 2.4.3;

Theorem 2.4.3:(Daniel *et al.*, 2022)

Let H and K be Hilbert spaces and let $O_{A \otimes B, C \otimes D}$ be basic elementary operator on $B(H \otimes K)$. If $\forall X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$, $A, C \in B(Z)$, $B, D \in B(K)$ $\zeta \in W_0(C)$, $\xi \in W_0(D)$ then we have;

$$\|O_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| = \text{Sup}_{\zeta \in W_0(C)} \text{Sup}_{\xi \in W_0(D)} \{|\zeta| |\xi| \|A\| \|B\|\}.$$

Finally, Daniel *et al.*, (2023) determined the bounds of the norm of elementary operator of length two in tensor product using the Stampfli maximal numerical range and obtained theorem 2.4.4

Theorem 2.4.4: (Daniel *et al.*, 2023).

Let H and K be Hilbert spaces and let $O_{A \otimes B, C \otimes D}$ be the elementary operator of length two on $B(H \otimes K)$. If $\forall X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$, $A_i, C_i \in B(H)$, $B_i, D_i \in B(K)$, $\zeta_i \in W_0(C_i)$, $\xi_i \in W_0(D_i) \forall i = 1, 2$ then we have;

$$\|\sum_{i=1}^n O_{2A_i \otimes B_i, C_i \otimes D_i} \setminus B(H \otimes K)\| = \text{Sup}_{\zeta_i \in W_0(C_i)} \text{Sup}_{\xi_i \in W_0(D_i)} \{\sum_{i=1}^n |\xi_i| |\zeta_i| \|A_i\| \|B_i\|\}$$

From the above definitions from Muiruri *et al.* (2018) on elementary operator in tensor product and the norms in Muiruri *et al.* (2018) and Daniel *et al.* (2022) created a better background for the extension of the determination of norms of elementary operator of finite length (n) and Jordan elementary operator in tensor product of C^* -algebras which this research solved in detail.

2.5 Tensor Product of C^* -algebras and Operator Spaces

This section discusses tensor products, the construction of tensor products, tensor products in Hilbert spaces, tensor products of C^* -algebras, and tensor product of operators. Finally, it looks at some properties of tensor products

2.5.1 Tensor product of Hilbert spaces.

Let H and K be fixed Hilbert spaces with inner/scalar products $\langle x, x \rangle$ and $\langle y, y \rangle$ respectively. The *tensor product* of H and K is a Hilbert space $H \otimes K$, where $\otimes : H \times K \rightarrow H \otimes K$, $(x, y) \rightarrow (x \otimes y)$ is a bilinear mapping such that;

- (i) The set of all vectors $(x \otimes y) : x \in H, y \in K$ form a total subset of $H \otimes K$, that is it's closed linear span(the minimum closed set containing the

intersection of all subspaces containing the set) is equal to $H \otimes K$.

NB: A subset T of a topological space V is a total fundamental set if the linear span of T is dense in V .

$$(ii) \langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle \quad \forall x_1, x_2 \in H \text{ and } \forall y_1, y_2 \in K.$$

In condition above putting $x_1 = x_2 = x$ and $y_1 = y_2 = y$, we have

$$\|x \otimes y\| = \|x\| \|y\|.$$

The linearity of the mapping $\otimes(x, y) = x \otimes y$ implies that \otimes is linear with respect to the two coordinates, that is;

$$(i) (x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y.$$

$$(ii) (\lambda x) \otimes y = \lambda(x \otimes y).$$

$$(iii) x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2.$$

$$(iv) x \otimes (\lambda y) = \lambda(x \otimes y).$$

$$(v) \text{ The set of all vectors } \varphi(x, y), x \in H \text{ and } y \in K \text{ form a total subset of } H \otimes K.$$

2.5.2 Some Properties of Tensor Products.

Below the study highlights some of the properties of tensor products. We assume we are given a tensor product $(H \otimes K, \otimes)$, where the symbol \otimes in the second coordinate abbreviates the associated bi-linear mapping.

Proposition 2.5.2.1: (Barberian.,2013) If $\sum_{k=1}^n x_k \otimes y_k = 0$ and y_k are linearly independent then $x_1 = \dots = x_n = 0$.

Proposition 2.5.2.2: (Barberian.,2013) If ε is a complex vector space and $\alpha : H \otimes K \rightarrow \varepsilon$ is any linear mapping, then $\psi(x, y) = \alpha(x, y)$ defines a bi-linear mapping $\psi : H \times K \rightarrow \varepsilon$.

Lemma 2.5.2.3: (Barberian.,2013) If $\psi : H \times K \rightarrow \varepsilon$ is a bi-linear mapping, then every relation $\sum_{k=1}^n x_k \otimes y_k = 0$ implies

$$\sum_{k=1}^n \psi(x_k, y_k) = 0.$$

Proposition 2.5.2.4: (Barberian.,2013) If $\psi : H \times K \rightarrow \varepsilon$ is any linear mapping, there exists a unique linear mapping $\alpha : H \otimes K \rightarrow \varepsilon$ such that;

$$\psi(x, y) = \alpha(x \otimes y).$$

Theorem 2.5.2.5: (Barberian.,2013) If $(H \otimes K, \otimes)$ and $(H \bar{\otimes} K, \bar{\otimes})$ are any two tensor products of H and K , then there exists a unique bounded linear mapping $U : H \otimes K \rightarrow H \bar{\otimes} K$ such that $U(x \otimes y) = x \bar{\otimes} y$. This mapping is unitary and $\langle Uu, Vv \rangle = \langle u, v \rangle$ for $u, v \in H \otimes K$.

This properties of tensor product creates the basis of the study's solutions

2.5.3 Tensor Product Bases

Consider Hilbert spaces H and K with orthonormal bases ϕ_i and ν_i respectively. The following basic result shows how to build basis for $H \otimes K$.

Theorem 2.5.3.1(Barberian.,2013) Let H and K be Hilbert spaces with orthonormal basis ϕ_i and ν_i respectively then $\{\phi_i \otimes \nu_i\}$ is an orthonormal basis for $H \otimes K$.

2.5.4 Construction of $H \otimes K$

As earlier, if H and K be fixed Hilbert spaces, then the following property hold;

If $T : K \rightarrow H$ is conjugate- linear that is

$$T(ay_1 + by_2) = \bar{a}T(y_1) + \bar{b}Ty_2$$

$\forall a, b \in \mathbb{C}, y_1, y_2 \in K$ and \bar{a}, \bar{b} are complex conjugate of a, b respectively (briefly C -linear) and bounded, there exist a unique c -linear bounded $T : H \rightarrow K$ such that

$$\langle Tx, y \rangle = \langle Ty, x \rangle \quad (x \in H, y \in K).$$

Thus if we denote by B_C the set of all bounded C -linear mappings $T : K \rightarrow H$, more precisely $B_C(K, H)$, clearly B_C is a complex vector space, relative to the natural operations eg $(\lambda T)x = \lambda(Tx)$.

If $T \in B_C(K, H)$ so is $T \in B_C(K, H)$. The following remark is due to Barberian (2013) and helps in obtaining the propertis of adjoint operator.

Remark 2.5.4.1:(Barberian.,2013) If $S, T \in B_C(K, H)$ and λ is a complex, then;

$$(i) \quad T^{**} = T$$

$$(ii) \|T^*\| = \|T\|$$

$$(iii) (S + T)^* = S^\# + T^*$$

$$(iv) (\lambda T)^* = \lambda T^*$$

$$(v) \text{ if } A \in B(H) \text{ and } B \in B(H) \text{ are bounded linear operators then } (ATB)^* = B^*T^*A^*$$

Denote by J the set of all $T \in B_C(K, H)$ for which $\sum_{k \in K} \|Tf_k\| < +\infty$.

J does not depend on specific basis $\{f_k\}$, more precisely $J = J(K, H)$ and J is a Hilbert space and has properties required of the Hilbert space ℓ in the definition of tensor product. If $S, T \in J$, set $\langle S, T \rangle = \sum_{k \in K} \langle Sf_k, Tf_k \rangle = \sum_{j \in J} \langle S^*e_i, T^*e_i \rangle$ (the sum being absolutely convergent and independent of specific orthonormal basis). Thus if $S, T \in J(K, H)$. Then $\langle S, T \rangle = \langle S^*, T^\# \rangle$ (The right-hand side is given by above applied in $J(K, H)$) more

$$\langle T, T \rangle = \sum_i \|Tf_k\|^2 \geq 0.$$

If $T \in J$, we write $\|T\|_2 = \langle T, T \rangle^{\frac{1}{2}}$. The notation $\|T\|$ is reserved for bounded of T as an operator of K . $T \rightarrow T^\#$ effects a scalar-product preserving linear isomorphism of $J(K, H)$ onto $J(K, H)$. In particular

$$\|T^\#\|_2 = \|T\|_2$$

. Thus if $T \in B_C(K, H)$, then $Tf_i = \sum_i \lambda_{ij}e_i$ for suitable unique scalars λ_{ij} . The array (λ_{ij}) is called the matrix of T (relative to the given orthonormal bases). By parseval's identity $\|Tf_i\|^2 = \sum_i |\lambda_{ij}e_i|^2$ hence

$$\sum_i \|Tf_i\|^2 = \sum_i |\lambda_{ij}e_i|^2.$$

Thus $T \in J = J(K, H)$ iff $\sum_i |\lambda_{ij}e_i|^2 < +\infty$, ie its matrix is "Square-Summable".

Denote by M the set of all matrices (λ_{ij}) for which;

$$\sum_i |\lambda_{ij}e_i|^2 < +\infty.$$

Then following property holds:

The correspondence $T \rightarrow (\lambda_{ij})(T \in J)$, where (λ_{ij}) is the matrix of T related to the given orthonormal bases, is a linear isomorphism of J onto M , preserving scalar product. In particular J is a Hilbert space related to $(\langle S, T \rangle)$.

If $x \in H$ and $y \in K, T \in J$ then;

$$\langle x \otimes y, T \rangle = \langle x, Ty \rangle.$$

This concept was very important for solving all the three specific objective of this study.

Clearly, for $x, x_1, x_2 \in H$ and $y, y_1, y_2 \in K$.

$$(i) \quad (x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y, (\lambda x) \otimes y = \lambda(x \otimes y). \\ x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2, x \otimes (\lambda y) = \lambda(x \otimes y).$$

$$(ii) \quad \langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle \text{ hence } \|x \otimes y\|_2 = \|x\| \|y\|.$$

$$(iii) \quad (x \otimes y)^* = y \otimes x.$$

Denote by $J_0 = J_0(K, H)$ the set of all finite sums of elements $x \otimes y$ of $H \otimes K$ ($x \in H, y \in K$) . Since $(\lambda x) \otimes y = \lambda(x \otimes y)$, J_0 is a linear subspace of J and from (iii) above $T \rightarrow T^\#$ maps $J_0(K, H)$ onto $J_0(K, H)$.

2.5.5 Tensor Product of Operators

Lemma 2.5.5.1 (Berberian.,2013 [Lemma 11]) If $T \in J(H), A \in B(H), B \in B(H)$, then;

$$ATB^* \in J(K, H) \text{ and } \|ATB^*\|_2 \leq \|A\| \|B\| \|T\|_2.$$

The following properties holds:-

(i) For a fixed $A \in B(H), B \in B(K)$, the mapping $T \rightarrow ATB^*(T \in J)$ is a bounded linear mapping of the Hilbert space J onto J . We denote $A \otimes B$ thus $A \otimes B \in B(J)$.

(ii) If $A \in B(H), B \in B(K)$, there exists a unique $C \in B(H \otimes K)$ such that;

$$C(x \otimes y) = Ax \otimes By, \forall x \in H, y \in K,$$

namely $C = A \otimes B$.

(iii) With the obvious notations

$$(a) \quad (A_1 + A_2) \otimes B = A_1 \otimes B + A_2 \otimes B, (\lambda A) \otimes B = \lambda(A \otimes B). \\ A \otimes (B_1 + B_2) = A \otimes B_1 + A \otimes B_2, A \otimes (\lambda B) = \lambda(A \otimes B).$$

- (b) $I \otimes I = I$ (I the identity mapping on H and K ($H \otimes K$)).
- (c) $(A \otimes B)(C \otimes D) = AC \otimes BD$.
- (d) $(A \otimes B)^* = A^* \otimes B^*$.
- (e) $\|A \otimes B\| = \|A\|\|B\|$.
- (f) $A \otimes B$ is invertible iff A and B are both invertible, in which case $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.
- (g) if σ denotes spectrum, then $\sigma(A \otimes B) = \sigma(A)\sigma(B) = (\lambda\mu : \lambda \in \sigma(A), \mu \in \sigma(B))$.

CHAPTER THREE

RESEARCH METHODOLOGY

3.1 Study Site

The research was based at Chuka University Postgraduate Library.

3.2 General Approach

For a successful completion of this study, background of C*-algebras, tensor product of C*-algebras, knowledge of elementary operators, linear operators, operator theory and tensor products of vectors and their properties was required.

3.3 Technical Approach

Specifically, to achieve the objectives, the study required the following;

3.3.1 Norm of Finite Length Elementary Operator

The definition of finite length elementary operator in tensor product of C*-algebras according to Muiruri *et al.* (2018) was used to formulate and solve the results of this study's first objective. The definition states that if $H \otimes K$ is a tensor product of complex Hilbert spaces H and K and $B(H \otimes K)$ is a set of bounded linear operators on $H \otimes K$. Then:-

An elementary operator, $T_n : B(H \otimes K) \rightarrow B(H \otimes K)$, is defined as;

$$T_n(X \otimes Y) = \sum_{i=1}^n A_i \otimes B_i(X \otimes Y)C_i \otimes D_i, \forall X \otimes Y \in B(H \otimes K),$$

where $A_i \otimes B_i, C_i \otimes D_i$ are fixed elements of $B(H \otimes K)$ and $A_i, C_i \in B(H), B_i, D_i \in B(K)$

The definition also served as the foundation in determining the lower and upper bounds of the norm of the finite length elementary operator in the tensor product of C*-algebras.

The result of the norm of basic elementary operator in tensor product of C*-algebras using finite rank operators and properties of tensor product of C*-algebras by Muiruri *et al.* (2018) also acted as a backbone for achieving the results of this study. They found that if H and K are complex Hilbert spaces and $B(H \otimes K)$ is a set of bounded linear operators on $H \otimes K$, then $\forall X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$, we have $\|M_{A \otimes B, C \otimes D}\| = \|A\| \|B\| \|C\| \|D\|$, where A, C and B, D are fixed elements in $B(H)$ and $B(K)$ respectively.

They first showed that $\|M_{A \otimes B, C \otimes D}\| \leq \|A\| \|B\| \|C\| \|D\|$

By definition,

$$\|M_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| = \sup\{\|M_{A \otimes B, C \otimes D}(X \otimes Y)\| : X \otimes Y \in B(H \otimes K), \|X \otimes Y\| = 1\}.$$

$$\|M_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| \geq \{\|M_{A \otimes B, C \otimes D}(X \otimes Y)\| : X \otimes Y \in B(H \otimes K), \|X \otimes Y\| = 1\}$$

Then we have $\forall \varepsilon \geq 0$;

$$\|M_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| - \varepsilon < \{\|M_{A \otimes B, C \otimes D}(X \otimes Y)\| : X \otimes Y \in B(H \otimes K), \|X \otimes Y\| = 1\}$$

$$\|M_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| - \varepsilon < \|A \otimes B(X \otimes Y)C \otimes D\| = \|AXC \otimes BYD\|$$

since $\varepsilon \geq 0$ was arbitrarily taken then

$$\|M_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| \leq \|AXC \otimes BYD\| = \|AXC\| \|BYD\|$$

since $\|X\| = 1$ then :

$$\|AXC\| \leq \|A\| \|X\| \|C\| = \|A\| \|C\|$$

Thus;

$$\|AXC\| \leq \|A\| \|C\|$$

Also, since $\|Y\| = 1$, then:

$$\|BYD\| \leq \|B\| \|D\|$$

Thus :

$$\|M_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| \leq \|AXC\| \|BYD\| \leq \|A\| \|B\| \|C\| \|D\|$$

Thus;

$$\|M_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| \leq \|A\| \|B\| \|C\| \|D\|$$

Conversily, let there exist a sequence $\{e_n \otimes f_n\}$ of unit vectors in $H \otimes K$ for each $e_n \in H$ and $f_n \in K$ then for each $n \geq 1$, we have:

$$\begin{aligned} \|M_{A \otimes B, C \otimes D}(X \otimes Y)(e_n \otimes f_n)\| &\leq \|M_{A \otimes B, C \otimes D}(X \otimes Y)\| \|e_n \otimes f_n\| \\ &\leq \|M_{A \otimes B, C \otimes D}\| \|(X \otimes Y)\| \|e_n \otimes f_n\| = \|M_{A \otimes B, C \otimes D}\| \|X\| \|Y\| \|e_n\| \|f_n\| \end{aligned}$$

Hence

$$\begin{aligned}
\|M_{A \otimes B, C \otimes D}\| &\geq \|M_{A \otimes B, C \otimes D}(X \otimes Y)(e_n \otimes f_n)\| = \|A \otimes B(X \otimes Y)C \otimes D(e_n \otimes f_n)\| \\
&= \|A \otimes B(X \otimes Y)C \otimes D(e_n \otimes f_n)\| = \|A \otimes B(X \otimes Y)C e_n \otimes D f_n\| \\
&= \|A \otimes B(X C e_n \otimes Y D f_n)\| = \|A X C e_n \otimes B Y D f_n\| = \|A X C e_n\| \|B Y D f_n\|
\end{aligned}$$

Thus we obtain that;

$$\|M_{A \otimes B, C \otimes D}\| \geq \|A X C e_n\| \|B Y D f_n\|$$

Now, let $u, v : H \rightarrow \mathbb{R}^+$ be functionals

From above, choosing unit vectors x_1, x_2 and define the finite rank operators

$$\begin{aligned}
A &= u \otimes x_1, x_1 \in H, \|x_1\| = 1 \text{ by } A e_n = (u \otimes x_1) e_n = u(e_n) x_1 \text{ and} \\
C &= v \otimes x_2, x_2 \in H, \|x_2\| = 1 \text{ by } C e_n = (v \otimes x_2) e_n = v(e_n) x_2
\end{aligned}$$

We observe that the norm of A is

$$\begin{aligned}
\|A\| &= \sup\{\|(u \otimes x_1) e_n\| : e_n \in H, \|x_1\| = 1\} \\
&= \sup\{\|u(e_n) x_1\| : e_n \in H, \|x_1\| = 1\} \\
&= \sup\{|u(e_n)| \|x_1\| : e_n \in H, \|x_1\| = 1\} \\
&= \sup\{|u(e_n)| : e_n \in H\} = |u(e_n)|
\end{aligned}$$

That is $\|A\| = |u(e_n)|$ for any $e_n \in H$

Likewise using the concept above, the norm of C is $\|C\| = |v(e_n)|$ for any $e_n \in H$

Therefore, we have $\|A X C e_n\| = \|(u \otimes x_1) X (v \otimes x_2) e_n\|$

By definition of finite rank operator then:

$$\begin{aligned}
\|(u \otimes x_1) X (v \otimes x_2) e_n\| &= \|(u \otimes x_1) X (v(e_n) x_2)\| \\
&= \|(u \otimes x_1) X (v(e_n) x_2)\| = |v(e_n)| \|(u \otimes x_1) X (x_2)\| \\
&= |v(e_n)| \|u(X(x_2)) x_1\| = |v(e_n)| \|u(X(x_2))\| \|x_1\| = \|A\| \|C\|
\end{aligned}$$

Then this shows that;

$$\|A X C e_n\| = \|A\| \|C\|$$

Following the same steps as shown above then;

$$\|B Y D f_n\| = \|B\| \|D\|$$

Thus it is clear that :

$$\|M_{A \otimes B, C \otimes D}\| \|X\| \|Y\| \geq \|AXCe_n\| \|BYDf_n\| = \|A\| \|C\| \|B\| \|D\|$$

Thus:

$$\|M_{A \otimes B, C \otimes D}\| \geq \|A\| \|B\| \|C\| \|D\|$$

Then;

$$\|M_{A \otimes B, C \otimes D}\| = \|A\| \|B\| \|C\| \|D\|$$

As a consequence of the above, Muiruri *et al.* (2018) related the norm of basic elementary operator in tensor product and the usual norm in this elementary operator in C^* -algebra and found that if H and K be complex Hilbert spaces and $B(H \otimes K)$ be the set of bounded linear operators on $H \otimes K$, then $\forall X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$, we have $\|M_{A \otimes B, C \otimes D}\| = \|M_{A,C}\| \|M_{B,D}\|$, where $M_{A,C}$ and $M_{B,D}$ are basic elementary operators in $B(H)$ and $B(K)$ respectively. This result was also of importance for this study since from the definition basic elementary operator is an elementary operator for $n = 1$, the study generalized Muiruri *et al.* (2018)'s findings for $n = 1$ to any n . In order to state and prove the norm of the finite length elementary operator in the tensor product of C^* -algebras, Muiruri *et al.* (2018) work was highly consulted.

The study also used the result of Daniel *et al.* (2022) where they used the stampi maximal numerical range to determine the norm of basic elementary operator in a tensor product to work out the norm. Daniel *et al.* (2022) gave this study more properties of tensor product that was combined with the concept of finite rank operators to obtain the norm of the finite length elementary operator in the tensor product of C^* -algebras. In their result they obtained that if H and K be Hilbert spaces and let $O_{A \otimes B, C \otimes D}$ be basic elementary operator on $B(H \otimes K)$. If $\forall X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$, $A, C \in B(H)$, $B, D \in B(K)$ $\zeta \in W_0(C)$, $\xi \in W_0(D)$ then;

$$\|O_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| = \text{Sup}_{\zeta \in W_0(C)} \text{Sup}_{\xi \in W_0(D)} \{|\zeta| |\xi| \|A\| \|B\|\}$$

The study also highly used the concept of Kawira *et al.* (2018). For their result they determined the norm of finite length elementary operator in C^* -algebra and found that for a complex Hilbert space H and algebra of bounded linear operator $B(H)$ on H . Then $\forall X \in B(H)$ with $\|X\| = 1$, we have $X(f) = f$ for all unit vector $f \in H$ $\|E_n\| = \sum_{i=1}^n \|A_i\| \|B_i\|$,

$n \in \mathbb{N}$. This result was utilized to establish the norm of the finite length elementary operator in the tensor product of C^* -algebras since the study introduce the idea of tensor product of C^* -algebras rather than C^* -algebras.

The definition of finite rank operator according to Erdman (2015) played a critical role in obtaining the norm of finite length operator mapping in this study. In his study he defined finite rank operator as; For a complex Hilbert space H , with dual H^* , define a functional $u : H \rightarrow \mathbb{R}$, then a finite rank operator $u \otimes x : H \rightarrow H$ is defined by $(u \otimes x)y = u(y)x, \forall y \in H$, where $u \in H^*$ and $x \in H$ is a unit vector, with;

$$\begin{aligned} \|u \otimes x\| &= \sup\{\|(u \otimes x)y\| : y \in H : \|x\| = 1\} \\ &= \sup\{\|u(y)x\| : y \in H\} \\ &= \sup\{|u(y)| \|x\| : y \in H : \|x\| = 1\} \\ &= |u(y)| : y \in H. \end{aligned}$$

The Concept of tensor product of Hilbert spaces and some properties of tensor products in Barberian (2013) was also of importance for this study. This entailed that for fixed Hilbert spaces $H = \{x, \dots\dots\}$ and $K = \{y, \dots\dots\}$ with inner/scalar products $\langle x, x \rangle$ and $\langle y, y \rangle$ respectively. The *tensor product* of H and K is a Hilbert space $H \otimes K$, where $\otimes : H \times K \rightarrow H \otimes K, (x, y) \rightarrow (x \otimes y)$ is a bilinear mapping such that;

- (i) The set of all vectors $(x \otimes y) : x \in H, y \in K$ form a total subset of $H \otimes K$, that is it's closed linear span(the minimum closed set containing the intersection of all subspaces containing the set) is equal to $H \otimes K$.

NB: A subset T of a topological space V is a total fundamental set if the linear span of T is dense in V .

- (ii) $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle \forall x_1, x_2 \in H$ and $\forall y_1, y_2 \in K$.

In condition above putting $x_1 = x_2 = x$ and $y_1 = y_2 = y$, we have

$$\|x \otimes y\| = \|x\| \|y\|.$$

The linearity of the mapping $\otimes(x, y) = x \otimes y$ implies that \otimes is linear with respect to the two coordinates, that is;

- (a) $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$.

$$(b) (\lambda x) \otimes y = \lambda(x \otimes y).$$

$$(c) x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2.$$

$$(d) x \otimes (\lambda y) = \lambda(x \otimes y).$$

(e) The set of all vectors $\varphi(x, y), x \in H$ and $y \in K$ form a total subset of $H \otimes K$.

Some of the properties of tensor products utilized in this research includes that for a tensor product $(H \otimes K, \otimes)$, where the symbol \otimes in the second coordinate abbreviates the associated bi-linear mapping with the obvious notations

$$(i) (A_1 + A_2) \otimes B = A_1 \otimes B + A_2 \otimes B, (\lambda A) \otimes B = \lambda(A \otimes B).$$

$$A \otimes (B_1 + B_2) = A \otimes B_1 + A \otimes B_2, A \otimes (\lambda B) = \lambda(A \otimes B).$$

$$(ii) I \otimes I = I \text{ (} I \text{ the identity mapping on } H \text{ and } K \text{ (} H \otimes K \text{))}.$$

$$(iii) (A \otimes B)(C \otimes D) = (AC \otimes BD).$$

$$(iv) (A \otimes B)^* = A^* \otimes B^*.$$

$$(v) \|A \otimes B\| = \|A\| \|B\|.$$

$$(vi) A \otimes B \text{ is invertible iff } A \text{ and } B \text{ are both invertible, in which case } (A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

$$(vii) \text{ if } \sigma \text{ denotes spectrum, then } \sigma(A \otimes B) = \sigma(A)\sigma(B) = (\lambda\mu : \lambda \in \sigma(A), \mu \in \sigma(B)).$$

The concept of tensor product and its properties played a crucial role in determining the norm of the finite length elementary operator in the tensor product of C^* -algebras in this study.

The following results and properties were also important in determining the norm of finite length elementary operator in tensor product of C^* -algebras as highlighted below;

(a) The work of Odero *et al.* (2015) where they determined the norm of tensor product elementary operator and found that for an inner derivation ζ_T , then

$$\|\zeta_{A,B}\| = \sup\{\|AX - XB\| : X \in B(H), \|X\| = 1\}$$

$$= \inf\{\lambda \in \mathbb{C} : A - \lambda + B - \lambda\} \text{ } A, B \in B(H).$$

This result was used to relate to the norm of finite length elementary operator in tensor product of C^* -algebras since both inner derivation and finite length elementary operator are types of elementary operator.

(b) Timoney (2001) in determining the general norm of basic elementary operator in C^* -algebra. Timoney (2001) obtained that for a C^* -algebra $B(H)$, then $\|M_{A,B}\| = \sup\{\|M_{A,B}(X)\| : X \in B(H)\} = \sup\{\|AXB\| : X \in B(H)\}$ where X denotes the set of unitaries in $B(H)$ was used to establish the background for this study's determination of the upper bound of the norms of finite length elementary operator in the tensor product of C^* -algebras.

(c) The concept of triangular inequality for norms of linear operators $S, T \in B(X, Y)$, of which $\forall x \in X : \|(S + T)(x)\| \leq (\|S\| + \|T\|)\|x\|$ was also used in determining the bounds of the norms of finite length elementary operator in the tensor product of C^* -algebras.

(d) Okelo (2021) determined the norm of sum of tensor product of operators where he found that, if $A, B \in B(H)$ and let $A \otimes B$ denote the tensor product of A and B then

$$\|A \otimes B + B \otimes A\| \leq \sqrt{2\|A\|^2\|B\|^2 + 2\|B^*A\|}.$$

In deed we have

$$\begin{aligned} \|A \otimes B + B \otimes A\|^2 &= \langle A \otimes B + B \otimes A, A \otimes B + B \otimes A \rangle \\ &= \langle A \otimes B, A \otimes B \rangle + \langle A \otimes B, B \otimes A \rangle + \langle B \otimes A, A \otimes B \rangle + \langle B \otimes A, B \otimes A \rangle \\ &= \langle A, A \rangle + \langle B, B \rangle + \langle A, B \rangle + \langle B, A \rangle + \langle B, B \rangle + \langle B, A \rangle + \langle A, B \rangle + \langle A, A \rangle + \langle B, B \rangle \\ &= \|A\|^2\|B\|^2 + \|B\|^2\|A\|^2 + 2\operatorname{Re}\langle B, A \rangle \langle A, B \rangle \end{aligned}$$

So by Cauchy-Schwartz inequality

$$\|A \otimes B + B \otimes A\|^2 \leq \|A\|^2\|B\|^2 + \|B\|^2\|A\|^2 + 2\operatorname{Re}\langle B, A \rangle \langle A, B \rangle = 2\|A\|^2\|B\|^2 + 2\|A\|^2\|B\|^2\|B\|^2\|A\|^2$$

But $\|B\| = \|B^*\|$, so the right hand of the above equation, we get:

$$\|A \otimes B + B \otimes A\|^2 \leq 2\|A\|^2\|B\|^2 + 2\|A\|^2\|B^*\|^2$$

Taking the positive square root on both sides yields.

$$\|A \otimes B + B \otimes A\| \leq \sqrt{\{2\|A\|^2\|B\|^2 + 2\|A\|^2\|B^*\|^2\}},$$

hence shown.

This gave this study an overview of how to obtain the norms and also gave the study properties appropriate when determining the norm of finite length elementary operator in tensor product of C^* -algebras.

(e) The Okelo and Agure (2011), theorem 4.2, where they determined the norm of the basic elementary operator using the finite rank operator and obtained that if H is a Hilbert space, $B(H)$ the algebra of bounded linear operators on H and if $M_{T,S} : B(H) \rightarrow B(H)$ is defined by $M_{T,S} = TXS, \forall X \in B(H)$ where A and B are fixed in $B(H)$ then;

$$\|M_{T,S}\| = \|T\|\|S\|$$

for $\|W\| = 1$ and for $W(x) = x, \forall$ unit vectors $x \in H$.

Indeed, we have

Since $\|M_{T,S} \setminus B(H)\| = \sup\{\|M_{T,S}(W)\| : W \in B(H), \|W\| = 1\}$

We have, $\forall \varepsilon > 0$;

$$\|M_{T,S} \setminus B(H)\| - \varepsilon < \{\|M_{T,S}(W)\| : W \in B(H), \|W\| = 1\}$$

Therefore

$$\|M_{T,S} \setminus B(H)\| - \varepsilon < \|TWS\| \leq \|T\|\|S\|.$$

Letting $\varepsilon \rightarrow 0$ we obtain

$$\|M_{T,S} \setminus B(H)\| \leq \|T\|\|S\|$$

On the other hand, we have:

$$\|M_{T,S} \setminus B(H)\| \geq \|M_{T,S}(W)\|, \forall W \in B(H), \|W\| = 1$$

with:

$$\|M_{T,S} \setminus B(H)\| = \sup\{\|M_{T,S}(W)x\| : x \in H, \|x\| = 1\}$$

So, letting $T = u \otimes x_1, \forall x_1 \in H, \|x_1\| = 1$ and $S = v \otimes x_2, \forall x_2 \in H, \|x_2\| = 1$, we have:

$\|M_{T,S} \setminus B(H)\| \geq \|M_{T,S}(W)\|, \forall W \in B(H), \|W\| = 1$, with T, S fixed in $B(H)$.

$$\begin{aligned} &= \|((u \otimes x_1)W(v \otimes x_2))y\| = \|(u \otimes x_1)W(v(y)x_2)\| \\ &= \|(u \otimes x_1)(v(y)Wx_2)\| = |v(y)|\|(u \otimes x_1)W(x_2)\| \\ &|v(y)|\|uW(x_2)x_1\| = |v(y)|\|u(W(x_2))\|\|x_1\| = \|T\|\|S\| \end{aligned}$$

Obtaining:

$$\|M_{T,S} \setminus B(H)\| \geq \|T\| \|S\|$$

Hence, we obtain;

$$\|M_{T,S} \setminus B(H)\| = \|T\| \|S\|.$$

The statement assisted this study in developing the solutions while the proof using the concept of finite rank operator gave the study the methodology to use in determining the norm.

3.3.2 Norm of Jordan Elementary Operator

The study used other known norm of Jordan elementary operator in C^* -algebra for the statement and proof of this objective. The following is the highlight of the most important results which created the bases of this research in this objective.

(a) Baraa and Boumazgour (2001), where they showed that for a complex Hilbert space H and bounded linear operators T and S on H , if $0 \in W_S(T^*S) \cup W_T(S^*T)$, then;

$$\|U_{T,S}\| \geq \|T\| \|S\|.$$

This result was used in obtaining the lower bound of the norms of Jordan elementary operators in tensor product of C^* -algebras

Also Baraa and Boumazgour (2001) found that for a complex Hilbert space H and bounded linear operators T and S on H , if $\|T\| \|S\| \in W_S(T^*S) \cap W_T(S^*T)$, then;

$$\|U_{T,S}\| = 2\|T\| \|S\|.$$

for suppose $\|T\| \|S\| \in W_S(T^*S) \cap W_T(S^*T)$. Then $\|T\| \|S\| \in W_S(T^*S)$ and $\|T\| \|S\| \in W_T(S^*T)$, and therefore we can find two sequences $\{x_n\}_{n \rightarrow \infty}$ and $\{y_n\}_{n \rightarrow \infty}$ of unit vectors in H such that:

$$\lim_{n \rightarrow \infty} \langle S^*T x_n, x_n \rangle = \|T\| \|S\|, \lim_{n \rightarrow \infty} \|T x_n\| = \|T\| \text{ and}$$

$$\lim_{n \rightarrow \infty} \langle S T^* y_n, y_n \rangle = \|T\| \|S\|, \lim_{n \rightarrow \infty} \|T^* y_n\| = \|T\|$$

since $|\langle S^*T x_n, x_n \rangle| \leq \|T x_n\| \|S x_n\|$ and $|\langle S T^* y_n, y_n \rangle| = \|T^* y_n\| \|S^* y_n\|$, then

$$\lim_{n \rightarrow \infty} \|S x_n\| = \|S\| \text{ and } \lim_{n \rightarrow \infty} \|S^* y_n\| = \|S\|$$

For each $n \geq 1$, we have:

$$\begin{aligned}
& \|(U_{T,S}(x_n \otimes y_n))S^*y_n\|^2 = \|(T(x_n \otimes y_n)S + S(x_n \otimes y_n)TS^*y_n)\|^2 \\
& = \|T(x_n \otimes y_n)SS^*y_n + S(x_n \otimes y_n)TS^*y_n\|^2 = \|\langle SS^*y_n, y_n \rangle Tx_n + \langle TS^*y_n, y_n \rangle Sx_n\|^2 \\
& = \|\langle S^*y_n, S^*y_n \rangle Tx_n + \langle TS^*y_n, y_n \rangle Sx_n\|^2 = \|\|S^*y_n\|^2Tx_n + \langle TS^*y_n, y_n \rangle Sx_n\|^2 \\
& = \langle \|S^*y_n\|^2Tx_n + \langle TS^*y_n, y_n \rangle Sx_n, \|S^*y_n\|^2Tx_n + \langle TS^*y_n, y_n \rangle Sx_n \rangle \\
& = \|\|S^*y_n\|^2Tx_n\|^2 + 2Re\langle \langle TS^*y_n, y_n \rangle Sx_n, \|S^*y_n\|^2Tx_n \rangle + |\langle TS^*y_n, y_n \rangle|^2 \|Sx_n\|^2 \\
& = \|S^*y_n\|^4\|Tx_n\|^2 + \|S^*y_n\|^2 2Re\langle TS^*y_n, y_n \rangle \langle Sx_n, Tx_n \rangle + |\langle TS^*y_n, y_n \rangle|^2 \|Sx_n\|^2 \\
& = \|S^*y_n\|^4\|Tx_n\|^2 + \|S^*y_n\|^2 2Re\langle TS^*y_n, y_n \rangle \langle T^*Sx_n, x_n \rangle + |\langle TS^*y_n, y_n \rangle|^2 \|Sx_n\|^2
\end{aligned}$$

Now, we have:

$$\|(U_{T,S}(x_n \otimes y_n))S^*y_n\| \leq \|U_{T,S}(x_n \otimes y_n)\| \|S^*y_n\| \leq \|U_{T,S}\| \|S^*y_n\|$$

Therefore;

$$\|U_{T,S}\|^2 \|S^*y_n\|^2 \geq \|S^*y_n\|^4\|Tx_n\|^2 + \|S^*y_n\|^2 2Re\langle TS^*y_n, y_n \rangle \langle T^*Sx_n, x_n \rangle + |\langle TS^*y_n, y_n \rangle|^2 \|Sx_n\|^2$$

letting $n \rightarrow \infty$, we obtain;

$$\|U_{T,S}\|^2 \|S\|^2 \geq \|S\|^4\|T\|^2 + 2\|S\|^4\|T\|^2 + \|S\|^4\|T\|^2 = 4\|S\|^4\|T\|^2$$

That is;

$$\|U_{T,S}\|^2 \geq 4\|S\|^2\|T\|^2 \text{ and this implies that:}$$

$$\|U_{T,S}\| \geq 2\|T\|\|S\|$$

Clearly $\|U_{T,S}\| \leq 2\|T\|\|S\|$ and therefore we obtain;

$$\|U_{T,S}\| = 2\|T\|\|S\|.$$

The statement and proof of Baraa and Boumazgour (2001) was used in the statement and proof of the equality in the norm of Jordan elementary operators in tensor product of C^* -algebras.

(b) The work of Timoney (2003) where he determined the norm of an elementary operator on a C^* -algebra using the notion of matrix-valued numerical ranges and a kind of geometrical

mean for positive matrices and tracial geometric mean and obtained that for a two dimensional Hilbert space H and $A, B \in B(H)$. If $U_{A,B} = M_{A,B} + M_{B,A}$ then $\|U_{A,B}\| \geq \|A\|\|B\|$. Highlighted another method the study could use to determine norm of Jordan elementary operators in tensor product of C^* -algebras.

(c) Blanco *et al.* (2004) where they determined norm of a Jordan elementary operator on a C^* -algebra using the notion of matrix valued numerical ranges and a kind of geometrical mean for positive matrices and tracial geometric mean. They concluded that for a two dimensional complex Hilbert space H and an algebra of bounded linear operators $B(H)$ on H . If $M_{A,B} : B(H) \rightarrow B(H)$ is defined by $M_{A,B}(X) = AXB + BXA, \forall X \in B(H)$ where A and B are fixed in $B(H)$ and e_1, e_2 are orthonormal basis of H then $\|AXB + BXA\| = \|A\|\|B\|$. This work was used in statement of the theorem in the determining the norm of Jordan elementary operators in tensor product of C^* -algebras.

(d) The Concept of tensor product of Hilbert spaces, C^* -algebras, and operators and some properties of tensor products in Barberian (2013) as indicated in subsection 3.3.1 was also used to achieve the norm of Jordan elementary operators in tensor product of C^* -algebras.

(e) The definition of Jordan elementary operator in tensor product of C^* -algebras in Muiruri *et al.* (2018) where they established that *the Jordan elementary operator*, $U_{A \otimes B, C \otimes D} : B(H \otimes K) \rightarrow B(H \otimes K)$, is defined as;

$$U_{A \otimes B, C \otimes D}(X \otimes Y) = A \otimes B(X \otimes Y)C \otimes D + C \otimes D(X \otimes Y)A \otimes B, \\ \forall X \otimes Y \in B(H \otimes K),$$

$A \otimes B, C \otimes D$ being fixed elements of $B(H \otimes K)$, where $A, C \in B(H)$, the set of bounded linear operators on H and $B, D \in B(K)$, the set of bounded linear operators on K was the basis for calculating the norm of the Jordan elementary operator in the tensor product of C^* -algebras

(f) Finally, the concept of finite rank operator and it's application in determining the norm was used to determine the norm of Jordan elementary operator in tensor product of C^* -algebras as indicated in subsection 3.3.1.

3.3.3 Relationship of the Norm in Tensor Product of C^* -algebras and the Norm in C^* -algebra

To establish the relationship of the Norm in $B(H \otimes K)$ and $B(H)$ and $B(K)$ the concept of relating the solutions in section 3.3.1 and 3.3.2 of this study in the determination of the norm of finite length elementary operator and Jordan elementary operator in tensor product of C^* -algebras and the known norms in C^* -algebras as indicated in subsections 3.3.1 and 3.3.2 was used.

Also the concept used in relating the norm of basic elementary operator in tensor product of C^* -algebras and norm of basic elementary operator in C^* -algebra in Muiruri *et al.* (2018) were used to relate the norm of Jordan elementary operator in tensor product of C^* -algebras and norm of Jordan elementary operator in C^* -algebra. In their work Muiruri *et al.* (2028) obtained that for complex Hilbert spaces H and K and the set of bounded linear operators $B(H \otimes K)$ on $H \otimes K$. Then $\forall X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$, we have $\|M_{A \otimes B, C \otimes D}\| = \|M_{A,C}\| \|M_{B,D}\|$, where $M_{A,C}$ and $M_{B,D}$ are basic elementary operators in $B(H)$ and $B(K)$ respectively.

Recall that from Okelo and Agure (2011) theorem 4.2 we have that;

$$\|M_{A,C} \setminus B(H)\| = \|A\| \|C\|, \text{ while } \|M_{B,D} \setminus B(K)\| = \|B\| \|D\|$$

Now, we have:

$$\|M_{A \otimes B, C \otimes D}\| = \|A\| \|B\| \|C\| \|D\|$$

We can rearrange this as:

$$\|M_{A \otimes B, C \otimes D}\| = \|A\| \|C\| \|B\| \|D\|$$

Notice that $A, C \in B(H)$, while $B, D \in B(K)$.

Then substituting, we obtain;

$$\|M_{A \otimes B, C \otimes D}\| = \|M_{A,C}\| \|M_{B,D}\|$$

3.3.4 Norm of an Arbitrary Elementary Operator in Tensor Product of C*-algebras

For this section to determined the conditions under which the norm of an arbitrary elementary operator in tensor product of C*-algebras is expressible in terms of norms of its coefficient operators the study used;

(a) The concept of *maximal numerical range* of an operator T which is defined to be the set

$$W_0(T) = \{\lambda : \langle Tx_n, x_n \rangle \rightarrow \lambda, \text{ where } \|x_n\| = 1 \text{ and } \|Tx_n\| \rightarrow \|T\|\}.$$

was used to identify the condition under which the norm of an arbitrary elementary operator in tensor product of C*-Algebras is expressible in terms of norms of its coefficient operators.

(b) The knowledge of rank one operator for which if $y, z \in H$ are unit vectors then *rank one operator*, $y \otimes z \in B(H)$, is defined as $(y \otimes z)x = \langle x, z \rangle y$ for all unit vectors $x \in H$ was used in this section to identify the norm in C*-algebras.

(c) The relationship between inner product and norm which is generally defined as, $\langle x, x \rangle = \|x\|^2$ was used in this study to obtain the norm.

(d) Also, this section used the concept of the statement and proof of King'ang'i *et al* (2014) where they shown that for a complex Hilbert space H and algebra of all bounded linear operators $B(H)$ on H . If for an operator $W \in B(H)$ with $\|W\| = 1$, we have $W(X) = X$ for all unit vectors $x \in H$, then;

$$\|E_2\| = \sum_{i=1}^2 \|T_i\| \|S_i\|,$$

where E_2 is the elementary operator on $B(H)$.

Recall that $E_2 : B(H) \rightarrow B(H)$ is defined as $E_2(W) = T_1WS_1 + T_2WS_2$, $\forall W \in B(H)$ and $T_i, S_i \in B(H)$ for a fixed $i = 1, 2$. We have

$$\|E_2 \setminus B(H)\| = \sup\{\|E_2(W)\| : W \in B(H), \|W\| = 1\}$$

Resulting to, $\|E_2 \setminus B(H)\| \geq \|E_2(W)\|$, $\forall W \in B(H)$ with $\|W\| = 1$

So, $\forall \varepsilon \geq 0$, $\|E \setminus B(H)\| - \varepsilon < \|E_2(W)\| \forall W \in B(H)$ with $\|W\| = 1$

Therefore, $\|E_2 \setminus B(H)\| - \varepsilon < \sum_{i=1}^2 \|T_i\| \|S_i\|$.

Now, Letting $\varepsilon \rightarrow 0$, we obtain:

$$\|E_2 \setminus B(H)\| \leq \sum_{i=1}^2 \|T_i\| \|S_i\|$$

Next, we show that $\|E_2 \setminus B(H)\| \geq \sum_{i=1}^2 \|T_i\| \|S_i\|$

Since $\|E_2(W)\| = \sup\{\|(E_2(W))x\| : x \in H, \|x\| = 1\}$, then we have

$$\|E_2(W)\| \geq \|(E_2(W))x\| : x \in H, \|x\| = 1\}.$$

But $E_2(W)x = (T_1WS_1 + T_2WS_2)x$.

Now, let $u_i, v_i : H \rightarrow \mathbb{R}^+$ be functionals $\forall i = 1, 2$

From above, Choosing unit vectors $y, z \in H$ and define the rank one operators

$$T_i = u_i \otimes y, y \in H, \|y\| = 1 \text{ by } T_i x = (u_i \otimes y)x = u_i(x)y$$

and

$$S_i = v_i \otimes z, z \in H, \|z\| = 1 \text{ by } S_i z_n = (v_i \otimes z)x = v_i(x)z_2$$

We observe that the norm of A is

$$\begin{aligned} \|T_i\| &= \sup\{\|(u_i \otimes y)x\| : x \in H, \|x\| = 1\} \\ &= \sup\{\|u_i(x)y\| : x \in H, \|x\| = 1\} \\ &= \sup\{|u_i(x)| \|y\| : x \in H, \|x\| = 1\} \\ &= \sup\{|u_i(x)| : x \in H, \|x\| = 1\} = |u_i(x)| \end{aligned}$$

That is $\|T_i\| = |u_i(x)|$ for any $x \in H \|x\| = 1$, for $i = 1, 2$

Like wise , the norm of S_i , is $\|S_i\| = |v_i(x)|$ for any unit vector $x \in H$ with $\|x\| = 1$ for $i = 1, 2$

Therefore, $\forall x \in H$ with $\|x\| = 1$, we have

$$\begin{aligned} E_2(W)x &= (T_1WS_1 + T_2WS_2)x = (T_1WS_1)x + (T_2WS_2)x \\ &= (u_1 \otimes y)W(v_1 \otimes z)x + (u_2 \otimes y)W(v_2 \otimes z)x \\ &= (u_1 \otimes y)Wv_1(x)z + (u_2 \otimes y)Wv_2(x)z \\ &= v_1(x)(u_1 \otimes y)W(z) + v_2(x)(u_2 \otimes y)W(z) \\ &= v_1(x)u_1(W(z))y + v_2(x)u_2(W(z))y \end{aligned}$$

Since $\|E_2(W)\| = \sup\{\|(E_2(W))x\| : x \in H, \|x\| = 1\}$, we have

$$\begin{aligned}
\|E_2 \setminus B(H)\|^2 &= \|v_1(x)u_1(W(z))y + v_2(x)u_2(W(z))y\|^2 \\
&= \langle v_1(x)u_1(W(z))y + v_2(x)u_2(W(z))y, v_1(x)u_1(W(z))y + v_2(x)u_2(W(z))y \rangle \\
&= \langle v_1(x)u_1(W(z))y + v_2(x)u_2(W(z))y, v_1(x)u_1(W(z))y \rangle + \\
&\quad \langle v_1(x)u_1(W(z))y + v_2(x)u_2(W(z))y, v_2(x)u_2(W(z))y \rangle \\
&= \langle v_1(x)u_1(W(z))y, v_1(x)u_1(W(z))y \rangle + \langle v_2(x)u_2(W(z))y, v_1(x)u_1(W(z))y \rangle \\
&\quad + \langle v_1(x)u_1(W(z))y, v_2(x)u_2(W(z))y \rangle + \langle v_2(x)u_2(W(z))y, v_2(x)u_2(W(z))y \rangle \\
&= \|v_1(x)u_1(W(z))y\|^2 + v_2(x)u_2(W(z))v_1(x)u_1(W(z))\langle y, y \rangle + \\
&\quad v_1(x)u_1(W(z))v_2(x)u_2(W(z))\langle yy \rangle + \|v_2(x)u_2(W(z))y\|^2 \\
&= |v_1(x)|^2 |u_1(W(z))|^2 \|y\|^2 + v_2(x)u_2(W(z))v_1(x)u_1(W(z)) \\
&\quad + v_1(x)u_1(W(z))v_2(x)u_2(W(z)) + |v_2(x)|^2 |u_2(W(z))|^2 \|y\|^2 \\
&= \{|v_1(x)| |u_1(W(z))|\}^2 + v_2(x)u_2(W(z))v_1(x)u_1(W(z)) \\
&\quad + v_1(x)u_1(W(z))v_2(x)u_2(W(z)) + \{|v_2(x)| |u_2(W(z))|\}^2 \\
&= \{|v_1(x)| |u_1(W(z))|\}^2 + 2v_1(x)u_1(W(z))v_2(x)u_2(W(z)) + \\
&\quad \{|v_2(x)| |u_2(W(z))|\}^2.
\end{aligned}$$

Now, since $v_1(x)$, $u_1(W(z))$, $v_2(x)$ and $u_2(W(z))$ are positive real numbers, we have

$$|v_1(x)| = v_1(x) = S_1, |u_1(W(z))| = u_1(W(z)) = T_1, |v_2(x)| = v_2(x) = S_2 \text{ and } |u_2(W(z))| = u_2(W(z)) = T_2.$$

$$\text{Thus } \|E_2 \setminus B(H)\|^2 \geq \{|T_1| |S_1|\}^2 + 2\{|T_1| |S_1| |T_2| |S_2|\}^2 + \{|T_2| |S_2|\}^2$$

= $\{|T_1| |S_1| + |T_2| |S_2|\}^2$ and hence we have

$$\|E_2 \setminus B(H)\| \geq \{|T_1| |S_1| + |T_2| |S_2|\} = \sum_{i=1}^2 \|T_i\| \|S_i\|.$$

which follows;

$$\|E_2 \setminus B(H)\| \geq \sum_{i=1}^2 \|T_i\| \|S_i\|$$

Now this implies that:

$$\|E_2 \setminus B(H)\| = \sum_{i=1}^2 \|T_i\| \|S_i\|.$$

(e) King'ng'i (2017) where he determined the conditions on which the norm of an elementary operator of length two is expressible in terms of the norms of its coefficients operators was important for determining the conditions under which the norm of finite length elementary operator in tensor product of C*-algebras is expressible in terms of the norms of its coefficients operators which is the third objective of this study. The following results are the highlights of King'ng'i (2017).

King'ang'i (2017) [Corollary 3.2]

Let H be a complex Hilbert space and T_i, S_i be bounded linear operators on H . Let $0 \in W_{S_1}(S_1^*S_2) \cup W_{S_2}(S_1^*S_2)$. Then $\|E_2\| \geq \|T_1\| \|S_1\|$, where E_2 is the elementary operator of length two for $i = 1, 2$.

King'ang'i (2017) [Theorem 3.3]

Let H be a complex Hilbert space and T_i, S_i be bounded linear operators on H for $i = 1, 2$. Let E_2 be an elementary operator of length two if $\|T_1\| \|T_2\| \in W_{T_1}(T_2T_1^*)$ and $\|S_1\| \|S_2\| \in W_{S_2}(S_1^*S_2)$. Then $\|E_2\| = \sum_{i=1}^n \|T_i\| \|S_i\|$.

For Suppose that $\|T_1\| \|T_2\| \in W_{T_1}(T_2T_1^*)$ and $\|S_1\| \|S_2\| \in W_{S_2}(S_1^*S_2)$. Then there are two sequence $\{x_n\}_{n \rightarrow \infty}$ and $\{y_n\}_{n \rightarrow \infty}$ such that;

$$\lim_{n \rightarrow \infty} \langle (T_2T_1^*)x_n, x_n \rangle = \|T_1\| \|T_2\|, \lim_{n \rightarrow \infty} \|T_1^*x_n\| = \|T_1\| \text{ and}$$

$$\lim_{n \rightarrow \infty} \langle (S_1^*S_2)y_n, y_n \rangle = \|S_1\| \|S_2\| \text{ and } \lim_{n \rightarrow \infty} \|S_2y_n\| = \|S_2\|.$$

Since;

$$| \langle (T_2T_1^*)x_n, x_n \rangle | \leq \|T_1^*x_n\| \|T_2^*x_n\|$$

and

$$| \langle (S_1^*S_2)y_n, y_n \rangle | \leq \|S_2y_n\| \|S_1y_n\|,$$

then $\lim_{n \rightarrow \infty} \|T_2^*\| = \|T_2\|$ and $\lim_{n \rightarrow \infty} \|S_1y_n\| = \|S_1\|$.

For each $n \geq 1$, we have;

$$\begin{aligned} \|E_2(x_n \otimes y_n)T_1^*y_n\|^2 &= \|(S_1(x_n \otimes y_n)T_1 + S_2(x_n \otimes y_n)T_2)T_1^*y_n\|^2 \\ &= \|S_1(x_n \otimes y_n)T_1T_1^*y_n + S_2(x_n \otimes y_n)T_2T_1^*y_n\|^2 \\ &= \|\langle T_1T_1^*y_n, y_n \rangle S_1x_n + \langle T_2T_1^*y_n, y_n \rangle S_2x_n\|^2 \\ &= \|\langle T_1^*y_n, T_1^*y_n \rangle S_1x_n + \langle T_2T_1^*y_n, y_n \rangle S_2x_n\|^2 \\ &= \|\|T_1^*\|^2 S_1x_n + \langle T_2T_1^*y_n, y_n \rangle S_2x_n\|^2 \\ &= \langle \|T_1^*\|^2 S_1x_n + \langle T_2T_1^*y_n, y_n \rangle S_2x_n, \|T_1^*\|^2 S_1x_n + \langle T_2T_1^*y_n, y_n \rangle S_2x_n \rangle \\ &= \|T_1^*\|^2 \|S_1x_n\|^2 + 2Re\langle \langle T_2T_1^*y_n, y_n \rangle S_2x_n, \|T_1^*y_n\|^* S_1x_n \rangle + | \langle T_2T_1^*y_n, y_n \rangle |^2 \|S_2x_n\|^2 \end{aligned}$$

$$\begin{aligned}
&= \|T_1^*\|^4 \|S_1 x_n\|^2 + 2 \|T_1^* y_n\|^2 \operatorname{Re} \langle T_2 T_1^* y_n, y_n \rangle \langle S_2 x_n, S_1 x_n \rangle + |\langle T_2 T_1^* y_n, y_n \rangle|^2 \|S_2 x_n\|^2 \\
&= \|T_1^*\|^4 \|S_1 x_n\|^2 + 2 \|T_1^* y_n\|^2 \operatorname{Re} \langle T_2 T_1^* y_n, y_n \rangle \langle S_1^* S_2 x_n, x_n \rangle + |\langle T_2 T_1^* y_n, y_n \rangle|^2 \|S_2 x_n\|^2.
\end{aligned}$$

Now, $\|\sum_{i=1}^2 M_{T_i, S_i}\| \|T_1\| \geq \|(\sum_{i=1}^2 M_{T_i, S_i}(x_n \otimes y_n)) T_1^* x_n\|$,

Therefore,

$$\|\sum_{i=1}^2 M_{T_i, S_i}\|^2 \|T_1\|^2 \geq \|T_1^*\|^4 \|S_1 x_n\|^2 + 2 \|T_1^* y_n\|^2 \operatorname{Re} \langle T_2 T_1^* y_n, y_n \rangle \langle S_1^* S_2 x_n, x_n \rangle + |\langle T_2 T_1^* y_n, y_n \rangle|^2 \|S_2 x_n\|^2$$

Letting $n \rightarrow \infty$, we obtain;

$$\|\sum_{i=1}^2 M_{T_i, S_i}\|^2 \|T_1\|^2 \geq \|T_1\|^4 \|S_1\|^2 + 2 \|T_1\|^2 \|T_2\| \|T_1\| \|S_1\| \|S_2\| + \|T_2\|^2 \|T_1\|^2 \|S_2\|^2.$$

That is:

$$\|\sum_{i=1}^2 M_{T_i, S_i}\|^2 \geq \|T_1\|^2 \|S_1\|^2 + 2 \|T_1\| \|T_2\| \|S_1\| \|S_2\| + \|T_2\|^2 \|S_2\|^2$$

$$\text{But } \|T_1\|^2 \|S_1\|^2 + 2 \|T_1\| \|T_2\| \|S_1\| \|S_2\| + \|T_2\|^2 \|S_2\|^2 = (\|T_1\| \|S_1\| + \|T_2\| \|S_2\|)^2$$

$$\text{Thus, } \|\sum_{i=1}^2 M_{T_i, S_i}\| \geq \|T_1\| \|S_1\| + \|T_2\| \|S_2\| = \sum_{i=1}^2 \|T_i\| \|S_i\|$$

$$\text{This follows } \|\sum_{i=1}^2 M_{T_i, S_i}\| \geq \sum_{i=1}^2 \|T_i\| \|S_i\|$$

$$\text{Clearly, } \|\sum_{i=1}^2 M_{T_i, S_i}\| \leq \sum_{i=1}^2 \|T_i\| \|S_i\|, \text{ and thus } \|\sum_{i=1}^2 M_{T_i, S_i}\| = \sum_{i=1}^2 \|T_i\| \|S_i\|$$

Hence;

$$\|E_2\| = \sum_{i=1}^2 \|T_i\| \|S_i\|.$$

This result of King'ang'i (2017) [Theorem 3.3] was used to clearly determine the conditions under which the norm of an arbitrary finite length elementary operator in tensor product of C^* -algebras is expressible in terms of its coefficient operators. .

The following results and properties helped in the construction and forming the basis of our expected results. They also helped in verifying, comparing and proving this study's results.

(i) Cauchy-Schwarz Inequality by Kreyszig (1991)

If X be an inner product space. Then for all $x, y \in X$ we have $|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$. Equality holds if and only if x and y are linearly dependent. Thus the Cauchy-Schwarz Inequality can be written as

$$\langle x, y \rangle \leq \|x\| \|y\|$$

In deed if $y = 0$, then $\langle x, y \rangle = 0$ and $\langle y, y \rangle = 0$ and therefore the inequality above is satisfied.

If $y \neq 0$ and for $\lambda = \frac{\langle x, y \rangle}{\langle y, y \rangle}$ then

$$\begin{aligned} \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} &= \frac{\langle x, y \rangle \overline{\langle x, y \rangle}}{\langle y, y \rangle} \\ &= \lambda \overline{\langle x, y \rangle} \\ &= \lambda \langle y, x \rangle. \end{aligned}$$

Since $\lambda \langle x, y \rangle = \overline{\lambda} \langle x, y \rangle$ then

$$\begin{aligned} \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} &= \lambda \langle y, x \rangle \\ &= \overline{\lambda} \langle x, y \rangle \\ &= |\lambda|^2 \langle y, y \rangle \end{aligned}$$

since $\langle x, y \rangle = \lambda \langle y, y \rangle$

Therefore by the axioms of the inner product of the inner product, then

$$\begin{aligned} 0 &\leq \langle x - \lambda y, x - \lambda y \rangle \\ &\leq \langle x, x \rangle + \langle x, -\lambda y \rangle + \langle -\lambda y, x \rangle + \langle -\lambda y, -\lambda y \rangle \end{aligned}$$

But $\overline{\lambda} \langle x, y \rangle = \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle}$ and $|\lambda|^2 \langle y, y \rangle = \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle}$ then;

$$\begin{aligned} 0 &\leq \langle x, x \rangle - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} + \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} \\ &\leq \langle x, x \rangle - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle}. \end{aligned}$$

This implies that $|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$.

The statement and proof of Cauchy-Schwarz Inequality helped in formulation of the proves in this study because of the concept of the relationship between inner product and the norm.

(ii) Parallelogram law by Ponnusamy and Silverman (2006)

For any element belonging to an inner product space X then

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

By simply expanding on the left hand side then the right hand side obtained

$$\begin{aligned}\|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle.\end{aligned}$$

Also

$$\begin{aligned}\|x - y\|^2 &= \langle x - y, x - y \rangle \\ &= \langle x, x \rangle + \langle x, -y \rangle + \langle -y, x \rangle + \langle -y, -y \rangle \\ &= \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle.\end{aligned}$$

This implies that

$$\begin{aligned}\|x + y\|^2 + \|x - y\|^2 &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle + \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle \\ &= 2\langle x, x \rangle + 2\langle y, y \rangle \\ &= 2(\|x\|^2 + \|y\|^2)\end{aligned}$$

The Parallelogram law was helpful for this study in bringing out clearly the concept of the norm and squares norm.

(iii) Cauchy-Schwarz-Bunyakovskii inequality by Gunawan (2002)

In every complex inner product space $(V, \langle \cdot, \cdot \rangle)$ equipped with the norm $\|u\| = \sqrt{\langle u, u \rangle} \forall u, v \in V$ we have $|\langle u, v \rangle| \leq \|u\| \|v\|$ and the equality holds if and only if u, v are linearly dependent where $\|\cdot\|$ is defined above.

The Cauchy-Schwarz-Bunyakovskii inequality helps in obtaining the upper bounds of the norms of elementary operators in this study.

(iv) The Ponnusamy and Silverman (2006) on norm on the inner product space.

The function $\| \cdot \| : X \rightarrow \mathbb{R}$ defined by $\|x\|_2 = \langle x, x \rangle$ is an norm on the inner product space X . Indeed

- (a) $\|x\| \geq 0$ for all $x \in X$.
- (b) $\|x\| = 0, x = o$ for all $x \in X$.
- (c) $\|\lambda x\|^2 = |\lambda|^2 \|x\|^2$ for all $x \in X$ and λ is a scalar. This implies that $\|\lambda x\| = |\lambda| \|x\|$.
- (d) $\|x + y\|^2 = \langle x + y, x + y \rangle$
 $= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$
 $= \|x\|^2 + 2\text{Re}\langle x, y \rangle + \|y\|^2$
 $\leq \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2 \leq (\|x\| + \|y\|)^2$

Hence

$$\|x + y\|^2 \leq (\|x\| + \|y\|)^2.$$

Getting the positive square root on both sides of the inequality above, then;

$$\|x + y\| \leq \|x\| + \|y\|.$$

This was very important for our study in bring out clearly the concept of the norm and inner product.

(v) Jordan-Von Neumann Theorem by Ponnusamy and Silverman (2006).

The norm of normed space X is given by an inner product space if and only if the norm satisfies the parallelogram law.

(vi) The Kinyanjui *et al.*,2018 (Lemma 2.8) on estimating the norms of norm-attainable elementary operators helped in estimating the norm of an operator. Which states that; If X, Y and Z be normed spaces over a field K . Then the following facts holds:

- (a) $B(X, Y)$ is a normed space over K under the operator norm.
- (b) If Y is complete, then $B(X, Y)$ is complete.

- (c) If Z is normed space over K and $T \in B(X, Z)$ and $S \in B(Y, Z)$, then the composition $S \circ T = ST$ belongs to $B(X, Z)$ and the operator norm is sub multiplicative, that is $\|S \circ T\| = \|ST\|$.
- (d) If X is finite dimensional, then any linear mapping bounded, that is $L(X, Y) = B(X, Y)$.

In deed if $T, S \in B(X, Y)$ then;

- (a) $\|T\| \geq 0$ since it is defined as the supremum of a set of non negative real numbers.
- (b) Also

$$\begin{aligned}\|\lambda T\| &= \sup_{\|x\|=1} \|(\lambda T)x\|. \\ &= \sup_{\|x\|=1} |\lambda| \|Tx\|. \\ &= |\lambda| \|T\|.\end{aligned}$$

for all scalar $\lambda \in \mathbb{F}$. In particular, $T \in B(X, Y)$ this implies that $\lambda T \in B(X, Y)$ for $\lambda \in \mathbb{F}$

- (c) Since the

$$\sup_{\|x\|=1} \|(T + S)x\| \leq \sup_{\|x\|=1} \|Tx\| + \sup_{\|x\|=1} \|Sx\|.$$

It follows

$$\|T + S\| \leq \|T\| + \|S\|$$

In particular $T, S \in B(X, Y)$ implies that $T + S \in B(X, Y)$

Therefore $B(X, Y)$ is a normed space. Taking T_n to be a Cauchy sequence in $B(X, Y)$ where Y is a Banach space, then

$$\|T_n - T_m\| \rightarrow 0 \text{ as } n, m \rightarrow \infty$$

since for each $x \in X$ we have

$$\begin{aligned} \|T_n x - T_m x\| &= \|(T_n - T_m)x\| \\ &\leq \|T_n - T_m\| \|x\| \rightarrow 0 \text{ as } n, m \rightarrow \infty \end{aligned}$$

it follows that T_n is a Cauchy sequence in Y for each $x \in X$. Thus by completeness of Y it has a limit in Y , that is

$$\lim_{n \rightarrow \infty} T_n x = T x$$

By the estimate $\|T_n\| - \|T_m\|$ the real sequence $\|T_n\|$ is Cauchy and hence bounded since all Cauchy sequences are bounded and therefore there exists a constant K with $\|T_n x\| \leq K \|x\|$ for all $x \in X$ and $n \geq 1$. $T_n \rightarrow T$ in the operator norm of $B(X, Y)$ since

$$\begin{aligned} \|T_n - T\| &= \sup_{\|x\| \leq 1} \|T_n x - T x\| \\ &= \sup_{\|x\| \leq 1} \lim_{n \rightarrow \infty} \|T_n x - T_m x\| \\ &\leq \sup_{\|x\| \leq 1} \lim_{n \rightarrow \infty} \|T_n - T_m\| \|x\| \\ &\leq \lim_{n \rightarrow \infty} \|T_n - T_m\| \end{aligned}$$

Since T_n is Cauchy, it follows that $\|T_n - T\| \rightarrow 0$ as $n \rightarrow \infty$ and therefore $T_n \rightarrow T$ is the operator norm of $B(X, Y)$. The operator norm is sub multiplicative since if $x \in X$, then

$$\begin{aligned} \|(ST)x\| &= \|S(Tx)\| \\ &\leq \|S\| \|Tx\| \\ &\leq \|S\| \|T\| \|x\|. \end{aligned}$$

Which follows that if $T \in B(X, Y)$ and $S \in B(X, Y)$ then $S, T \in B(X, Y)$ and

$$\|S \circ T\| \leq \|S\| \|T\|.$$

(vii) The King'ang'i (2018) (Theorem 3.4) on the conditions under which the norm of elementary operator of length two is expressible in terms of the norm of its co-efficient operators was also used to arrive to our results in this study. King'ang'i (2018) states that;

If E_2 is an elementary operator of length two on $B(H)$ and $S_1, S_2 \in B(H)$, then if $\lambda_i \in W \circ S_i$ for every $\lambda_i \in \mathbb{C}$ $i = 1, 2$ then,

$$\|E_2\| \geq \sup_{\lambda_i \in W \circ (S_i)} \{ \|\sum_{i=1}^n \lambda_i T_i\| : T_i \in B(H) i = 1, 2 \}.$$

For the proof King'ang'i (2018) let $\{x_n\}_{(n>1)}$ be a sequence of vectors of length one in a complex Hilbert space H and Defined rank one operator, $(y \otimes x_n)(x) \in B(H)$ for a unit vector $y \in H$ as $(y \otimes x_n)x = \langle x, y_n \rangle y \quad \forall x \in H$.

Also Defined the $W_o(S_i) \quad \forall i = 1, 2$ of defined as

$$W_o(S_1) = \{ \lambda_1 \in \mathbb{C} : \langle S_1 x_n, x_n \rangle \rightarrow \lambda_1, \|x_n\| = 1, \|S_1 x_n\| \rightarrow \|S_1\| \}$$

and

$$W_o(S_2) = \{ \lambda_2 \in \mathbb{C} : \langle S_2 x_n, x_n \rangle \rightarrow \lambda_2, \|x_n\| = 1, \|S_2 x_n\| \rightarrow \|S_2\| \}$$

Now $\forall \lambda_i \in H$, if $\lambda_i \in W_o(S_i) \quad \forall S_i \in B(Z)$, $\forall i = 1, 2$ then a sequence $\{x_n\}_{(n>1)}$ of vectors of length one exists in H such that;

- (a) $\lim_{n \rightarrow \infty} \langle S_1 x_n, x_n \rangle = \lambda_1$
- (b) $\lim_{n \rightarrow \infty} \|S_1 x_n\| = \|S_1\|$
- (c) $\lim_{n \rightarrow \infty} \langle S_2 x_n, x_n \rangle = \lambda_2$
- (d) $\lim_{n \rightarrow \infty} \|S_2 x_n\| = \|S_2\|$

By finite rank one operator, the elementary operator of length two norm of is given as

$$\begin{aligned} \|(y \otimes x_n) \otimes x_n\| &= \|(\sum_{i=1}^2 M_{(T_i, S_i)}(y \otimes x_n))x_n\| \\ &= \|(\sum_{i=1}^2 M_{(T_i, S_i)}(y \otimes x_n))x_n\| \leq \| \sum_{i=1}^2 M_{T_i, S_i}(y \otimes x_n) \| \|x_n\| \leq \| \sum_{i=1}^2 M_{T_i, S_i} \| \|y\| \|x_n\| \|x_n\| \\ &\leq \| \sum_{i=1}^2 M_{(T_i, S_i)} \| \|y\| \|x_n\| \|x_n\| \leq \| \sum_{i=1}^2 M_{(T_i, S_i)} \| \end{aligned}$$

since the y is a unit vector and x_n is a unit sequence such that $\|y\| = 1, \|x_n\| = 1$ Therefore;

$$\begin{aligned} \left\| \sum_{i=1}^2 M_{T_i, S_i} \right\| &\geq \left\| \sum_{i=1}^2 M_{T_i, S_i}(y \otimes x_n)x_n \right\| \\ &\geq \left\| \sum_{i=1}^2 M_{T_i, S_i}(y \otimes x_n)x_n \right\| \\ &\geq \| (T_1)(y \otimes x_n)x_n(S_1) + (T_2)(y \otimes x_n)x_n(S_2) \| \\ &\geq \| \langle S_1 x_n, x_n \rangle T_1 y + \langle S_2 x_n, x_n \rangle T_2 m \| \end{aligned}$$

thus

$$\|\sum_{i=1}^2 M_{T_i, S_i, G_i}\| \geq \| \langle S_1 x_n, x_n \rangle T_1 y + \langle S_2 x_n, x_n \rangle T_2 m \|$$

by taking the limits both sides as $n \rightarrow \infty$ for the inequality and $\forall \lambda_i \in H$. If $\lambda_i \in W_o(S_i) \forall S_i \in B(H), \forall i = 1, 2$ then \exists a sequence $\{x_n\}_{(n>1)}$ of vectors of length one in h such that

$$(a) \lim_{n \rightarrow \infty} \langle S_1 x_n, x_n \rangle = \lambda_1$$

$$(b) \lim_{n \rightarrow \infty} \|S_1 x_n\| = \|S_1\|$$

$$(c) \lim_{n \rightarrow \infty} \langle S_2 x_n, x_n \rangle = \lambda_2$$

$$(d) \lim_{n \rightarrow \infty} \|S_2 x_n\| = \|S_2\|$$

then;

$$\|\sum_{i=1}^2 M_{T_i, S_i}\| \geq \|\lambda_1 T_1 y + \lambda_2 T_2 Y\| = \|\sum_{i=1}^2 \lambda_i T_i y\|$$

since the y is a unit vector such that $\|y\| = 1$. This is true for any $\lambda \in W_o(S)$ and for any unit vector $y \in H$. Since λ and the unit vector are chosen arbitrarily, then we get the, supremum for the lower bound;

$$\|\sum_{i=1}^2 M_{T_i, S_i}\| \leq \sup_{\lambda_i \in W_o(S_i)} \sup_{\|y\|=1} \|\sum_{i=1}^2 \lambda_i T_i y\| = \sup_{\lambda_i \in W_o(S_i)} \|\sum_{i=1}^2 \lambda_i T_i\|$$

3.4 Ethical Consideration

The proposal was approved by Chuka University's ethical review committee and the research permit obtained from National Commission for Science, Technology, and Innovation (NA-COSTI) before commencing the research. Ethical issues such as plagiarism, falsifying of the work, and biases was strictly observed. The outcomes of the study was carefully and critically reviewed so that the results obtained was credible and was reported with honesty and integrity. The results were published in a transparent manner in order to help further knowledge advancement in areas of pure mathematics. Finally, the results from this study upon publishing will not be retrieved or transmitted without the knowledge of Chuka University.

CHAPTER FOUR
RESULTS AND DISCUSSIONS

4.1 Introduction

From section 2.4, according to Muiruri *et al.* (2018), if $H \otimes K$ is a tensor product of complex Hilbert spaces H and K and $B(H \otimes K)$ is the set of bounded linear operators on $H \otimes K$, $A \otimes B, C \otimes D$ being fixed elements of $B(H \otimes K)$, where $A, C \in B(H)$, the set of bounded linear operators on H and $B, D \in B(K)$, the set of bounded linear operators on K , then:-
The Basic elementary operator, $M_{A \otimes B, C \otimes D} : B(H \otimes K) \rightarrow B(H \otimes K)$, is defined as:-

$$M_{A \otimes B, C \otimes D}(X \otimes Y) = A \otimes B(X \otimes Y)C \otimes D, \forall X \otimes Y \in B(H \otimes K).$$

An elementary operator, $T_n : B(H \otimes K) \rightarrow B(H \otimes K)$, is defined as;

$$T_n(X \otimes Y) = \sum_{i=1}^n A_i \otimes B_i(X \otimes Y)C_i \otimes D_i, \forall X \otimes Y \in B(H \otimes K),$$

$A_i \otimes B_i, C_i \otimes D_i$ being fixed elements of $B(H \otimes K)$, where $A_i, C_i \in B(H)$, is the set of bounded linear operators on H and $B_i, D_i \in B(K)$, the set of bounded linear operators on K .

When $n = 2$ then we obtain *an elementary operator of length two* which is defined as

$$\begin{aligned} T_2(X \otimes Y) &= A_1 \otimes B_1(X \otimes Y)C_1 \otimes D_1 + A_2 \otimes B_2(X \otimes Y)C_2 \otimes D_2 \\ &= \sum_{i=1}^2 A_i \otimes B_i(X \otimes Y)C_i \otimes D_i, \forall X \otimes Y \in B(H \otimes K) \end{aligned}$$

and $A_i \otimes B_i, C_i \otimes D_i$ being fixed elements of $B(H \otimes K)$, where $A_i, C_i \in B(H)$, the set of bounded linear operators on H and $B_i, D_i \in B(K)$, the set of bounded linear operators on K .

The Jordan elementary operator, $U_{A \otimes B, C \otimes D} : B(H \otimes K) \rightarrow B(H \otimes K)$, is defined as

$$\begin{aligned} U_{A \otimes B, C \otimes D}(X \otimes Y) &= A \otimes B(X \otimes Y)C \otimes D + C \otimes D(X \otimes Y)A \otimes B \\ &\quad \forall X \otimes Y \in B(H \otimes K), \end{aligned}$$

$A \otimes B, C \otimes D$ being fixed elements of $B(H \otimes K)$, where $A, C \in B(H)$, the set of bounded linear operators on H and $B, D \in B(K)$, the set of bounded linear operators on K .

Thus this can be extended to the definitions of other types of elementary operator to obtain;

- The left multiplication operator, $L_{A \otimes B} : B(H \otimes K) \rightarrow B(H \otimes K)$, is defined by

$$L_{A \otimes B}(X \otimes Y) = A \otimes B(X \otimes Y) \forall X \otimes Y \in B(H \otimes K),$$

$A \otimes B$, being fixed element of $B(H \otimes K)$. Where $A \in B(H)$, the set of bounded linear operators on H and $B \in B(K)$, the set of bounded linear operators in K .

- The right multiplication operator, $R_{C \otimes D} : B(H \otimes K) \rightarrow B(H \otimes K)$, is defined by

$$R_{C \otimes D}(X \otimes Y) = (X \otimes Y)C \otimes D; \forall X \otimes Y \in B(H \otimes K),$$

$C \otimes D$, being fixed element of $B(H \otimes K)$. Where $C, \in B(H)$, the set of bounded linear operators on H and $D \in B(K)$, the set of bounded linear operators in K .

- The general derivation operator is defined by;

$$\delta_{A \otimes B, C \otimes D} = (A \otimes B)(X \otimes Y) - (X \otimes Y)(C \otimes D) = L_{A \otimes B} - R_{C \otimes D},$$

where $L_{A \otimes B}, R_{C \otimes D}$ are left and right multiplication operators as defined above.

- the inner derivation operator is defined by;

$$\delta_{A \otimes B, A \otimes B} = (A \otimes B)(X \otimes Y) - (X \otimes Y)(A \otimes B) = L_{A \otimes B} - R_{A \otimes B}.$$

4.2 Norm of finite length Elementary Operator in Tensor Product of C^* -algebras

In this section, this study investigated the bounds of the norm of an elementary operator of finite length in a tensor product of C^* -algebras by stating and proving theorem 4.2.1

Theorem 4.2.1:

Let $H \otimes K$ be tensor product of complex Hilbert spaces H and K and $B(H \otimes K)$ be the set of bounded linear operators on $H \otimes K$. Then $\forall X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$, $\|T_n\| = \sum_{i=1}^n \|A_i\| \|B_i\| \|C_i\| \|D_i\|$, where T_n is the Elementary operator of finite length in a tensor product of C^* -algebras and A_i, C_i and B_i, D_i are in $B(H)$ and $B(K)$ respectively.

Proof. By definition;

$$\|T_n \setminus B(H \otimes K)\| = \sup\{\|T_n(X \otimes Y)\| : X \otimes Y \in B(H \otimes K), \|X \otimes Y\| = 1\}.$$

For any $T_n : B(H \otimes K) \rightarrow B(H \otimes K)$ and $X \otimes Y : H \otimes K \rightarrow H \otimes K$,

$$\|T_n(X \otimes Y)\| \leq \|T_n\| \|X \otimes Y\| \quad \forall X \otimes Y \in B(H \otimes K).$$

Or;

$$\|T_n\| \|X \otimes Y\| \geq \|T_n(X \otimes Y)\| \quad \forall X \otimes Y \in B(H \otimes K).$$

Letting the $\varepsilon \in (0, \|T_n\|)$ and therefore $\|T_n\| - \varepsilon > 0$.

Since $\|T_n\| = \sup\{\|T_n(X \otimes Y)\| : X \otimes Y \in B(H \otimes K), \|X \otimes Y\| = 1\}$,

then $\exists X' \otimes Y' \in B(H \otimes K)$ with $\|X' \otimes Y'\| \neq 0$ for which,

$$(\|T_n\| - \varepsilon) \|X' \otimes Y'\| < \|T_n(X' \otimes Y')\|.$$

That is;

$$\|T_n\| - \varepsilon < \frac{1}{\|X' \otimes Y'\|} \|T_n(X' \otimes Y')\|.$$

This implies that;

$$\|T_n\| - \varepsilon < \|T_n \frac{X' \otimes Y'}{\|X' \otimes Y'\|}\|.$$

Since $\frac{X' \otimes Y'}{\|X' \otimes Y'\|}$ is a unit in $B(H \otimes K)$ then;

$$\|T_n \setminus B(H \otimes K)\| - \varepsilon < \|T_n(X \otimes Y)\| \quad \text{where } X \otimes Y \in B(H \otimes K) \text{ with } \|X \otimes Y\| = 1.$$

Since ε was arbitrarily chosen then;

$$\|T_n \setminus B(H \otimes K)\| \leq \|T_n(X \otimes Y)\| \quad \text{where } X \otimes Y \in B(H \otimes K) \text{ with } \|X \otimes Y\| = 1.$$

Since $\|T_n(X \otimes Y)\| = \|\sum_{i=1}^n A_i \otimes B_i(X \otimes Y)C_i \otimes D_i\|$ then;

$$\|T_n \setminus B(H \otimes K)\| \leq \|\sum_{i=1}^n A_i \otimes B_i(X \otimes Y)C_i \otimes D_i\| = \|A_1 \otimes B_1(X \otimes Y)C_1 \otimes D_1 + A_2 \otimes B_2(X \otimes Y)C_2 \otimes D_2 + \dots + A_n \otimes B_n(X \otimes Y)C_n \otimes D_n\|, \|X \otimes Y\| = 1.$$

From properties of tensor of operators we have $(A \otimes B)(X \otimes Y) = AX \otimes BY$ then;

$$\begin{aligned} \|T_n \setminus B(H \otimes K)\| &\leq \|A_1XC_1 \otimes B_1YD_1 + A_2XC_2 \otimes B_2YD_2 + \dots + \\ &A_nXC_n \otimes B_nYD_n\| \leq \|A_1XC_1 \otimes B_1YD_1\| + \|A_2XC_2 \otimes B_2YD_2\| + \dots + \\ &\|A_nXC_n \otimes B_nYD_n\|, \|X \otimes Y\| = 1. \text{ (By triangular property)} \end{aligned}$$

Thus;

$$\begin{aligned} \|T_n \setminus B(H \otimes K)\| &\leq \|A_1XC_1 \otimes B_1YD_1\| + \|A_2XC_2 \otimes B_2YD_2\| + \dots + \\ &\|A_nXC_n \otimes B_nYD_n\|, \|X \otimes Y\| = 1. \end{aligned}$$

Then by property of tensor product $\|A \otimes B\| = \|A\|\|B\|$ then

$$\begin{aligned} \|T_n \setminus B(H \otimes K)\| &\leq \|A_1XC_1\|\|B_1YD_1\| + \|A_2XC_2\|\|B_2YD_2\| + \dots + \|A_nXC_n\|\|B_nYD_n\|, \\ \|X \otimes Y\| &= 1. \end{aligned}$$

Since $\|A_iXC_i\| \leq \|A_i\|\|X\|\|C_i\| = \|A_i\|\|C_i\|$, since $\|X\| = 1$ and

$$\|B_iYD_i\| \leq \|B_i\|\|Y\|\|D_i\| = \|B_i\|\|D_i\|, \text{ since } \|Y\| = 1.$$

Therefore;

$$\begin{aligned} \|T_n \setminus B(H \otimes K)\| &\leq \|A_1\|\|X\|\|C_1\|\|B_1\|\|Y\|\|D_1\| + \|A_2\|\|X\|\|C_2\|\|B_2\|\|Y\|\|D_2\| + \dots + \\ &\|A_n\|\|X\|\|C_n\|\|B_n\|\|Y\|\|D_n\|, \|X \otimes Y\| = 1 \text{ becomes;} \\ \|T_n \setminus B(H \otimes K)\| &\leq \|A_1\|\|C_1\|\|B_1\|\|D_1\| + \|A_2\|\|C_2\|\|B_2\|\|D_2\| + \dots + \|A_n\|\|C_n\|\|B_n\|\|D_n\|. \end{aligned}$$

Thus;

$$\|T_n\| \leq \sum_{i=1}^n \|A_i\|\|B_i\|\|C_i\|\|D_i\|. \quad (4.1)$$

Conversely, for $T_n : B(H \otimes K) \rightarrow B(H \otimes K)$ and $X \otimes Y : H \otimes K \rightarrow H \otimes K$, let there exist unit vector $(e \otimes f)$ of $H \otimes K$ where $e \in H$ and $f \in K$

Then;

$$\begin{aligned} \|T_n(X \otimes Y)(e \otimes f)\| &\leq \|T_n(X \otimes Y)\|\|e \otimes f\| \leq \|T_n\|\|X \otimes Y\|\|e \otimes f\| = \|T_n\|\|X\|\|Y\|\|e\|\|f\| = \\ &\|T_n\| \text{ since } \|X\| = \|Y\| = \|e\| = \|f\| = 1. \end{aligned}$$

This implies that:

$$\begin{aligned}
\|T_n\| &\geq \|T_n(X \otimes Y)(e \otimes f)\| = \|\{(A_1 \otimes B_1)(X \otimes Y)(C_1 \otimes D_1) + (A_1 \otimes B_1)(X \otimes Y)(C_1 \otimes D_1) + \dots + (A_n \otimes B_n)(X \otimes Y)(C_n \otimes D_n)\}(e \otimes f)\| \\
&= \|\{(A_1 \otimes B_1)(X \otimes Y)(C_1 \otimes D_1)\}(e \otimes f) + \{(A_2 \otimes B_2)(X \otimes Y)(C_2 \otimes D_2)\}(e \otimes f) + \dots + (A_n \otimes B_n)(X \otimes Y)(C_n \otimes D_n)(e \otimes f)\| \\
&= \|\{(A_1 \otimes B_1)(X \otimes Y)(C_1 e \otimes D_1 f)\} + \{(A_1 \otimes B_1)(X \otimes Y)(C_2 e \otimes D_2 f) + \dots + (A_n \otimes B_n)(X \otimes Y)(C_n e \otimes D_n f)\}\| \\
&= \|\{(A_1 \otimes B_1)(XC_1 e \otimes YD_1 f)\} + \{(A_2 \otimes B_2)(XC_2 e \otimes YD_2 f)\} + \dots + \{(A_n \otimes B_n)(XC_n e \otimes YD_n f)\}\| \\
&= \|(A_1 X C_1 e \otimes B_1 Y D_1 f) + (A_2 X C_2 e \otimes B_2 Y D_2 f) + \dots + (A_n X C_n e \otimes B_n Y D_n f)\|
\end{aligned}$$

Thus:

$$\|T_n\| \geq \|(A_1 X C_1 e \otimes B_1 Y D_1 f) + (A_2 X C_2 e \otimes B_2 Y D_2 f) + \dots + (A_n X C_n e \otimes B_n Y D_n f)\|$$

Squaring both sides we obtain;

$$\begin{aligned}
\|T_n\|^2 &\geq \|(A_1 X C_1 e \otimes B_1 Y D_1 f) + (A_2 X C_2 e \otimes B_2 Y D_2 f) + \dots + (A_n X C_n e \otimes B_n Y D_n f)\|^2 \\
&= \langle (A_1 X C_1 e \otimes B_1 Y D_1 f) + (A_2 X C_2 e \otimes B_2 Y D_2 f) + \dots + (A_n X C_n e \otimes B_n Y D_n f), \\
&\quad (A_1 X C_1 e \otimes B_1 Y D_1 f) + (A_2 X C_2 e \otimes B_2 Y D_2 f) + \dots + (A_n X C_n e \otimes B_n Y D_n f) \rangle \\
&= \langle (A_1 X C_1 e \otimes B_1 Y D_1 f), (A_1 X C_1 e \otimes B_1 Y D_1 f) + (A_2 X C_2 e \otimes B_2 Y D_2 f) + \dots + \\
&\quad + (A_n X C_n e \otimes B_n Y D_n f) \rangle + \langle (A_2 X C_2 e \otimes B_2 Y D_2 f), (A_1 X C_1 e \otimes B_1 Y D_1 f) + \\
&\quad (A_2 X C_2 e \otimes B_2 Y D_2 f) + \dots + (A_n X C_n e \otimes B_n Y D_n f) \rangle + \dots + \langle (A_n X C_n e \otimes B_n Y D_n f), \\
&\quad (A_1 X C_1 e \otimes B_1 Y D_1 f) + (A_2 X C_2 e \otimes B_2 Y D_2 f) + \dots + (A_n X C_n e \otimes B_n Y D_n f) \rangle \\
&= \langle (A_1 X C_1 e \otimes B_1 Y D_1 f), (A_1 X C_1 e \otimes B_1 Y D_1 f) \rangle + \langle (A_1 X C_1 e \otimes B_1 Y D_1 f), \\
&\quad (A_2 X C_2 e \otimes B_2 Y D_2 f) \rangle + \langle (A_1 X C_1 e \otimes B_1 Y D_1 f), (A_n X C_n e \otimes B_n Y D_n f) \rangle + \\
&\quad \langle (A_2 X C_2 e \otimes B_2 Y D_2 f), (A_1 X C_1 e \otimes B_1 Y D_1 f) \rangle + \langle (A_2 X C_2 e \otimes B_2 Y D_2 f), \\
&\quad (A_2 X C_2 e \otimes B_2 Y D_2 f) \rangle + \dots + \langle (A_2 X C_2 e \otimes B_2 Y D_2 f), (A_n X C_n e \otimes B_n Y D_n f) \rangle + \\
&\quad \langle (A_n X C_n e \otimes B_n Y D_n f), (A_1 X C_1 e \otimes B_1 Y D_1 f) \rangle + \langle (A_n X C_n e \otimes B_n Y D_n f), \\
&\quad (A_2 X C_2 e \otimes B_2 Y D_2 f) \rangle + \dots + \langle (A_n X C_n e \otimes B_n Y D_n f), (A_n X C_n e \otimes B_n Y D_n f) \rangle
\end{aligned}$$

Since $\langle X_1 \otimes Y_1, X_2 \otimes Y_2 \rangle = \langle X_1, X_2 \rangle \langle Y_1, Y_2 \rangle$

$$= \langle A_1 X C_1 e, A_1 X C_1 e \rangle \langle B_1 Y D_1 f, B_1 Y D_1 f \rangle + \langle A_1 X C_1 e, A_2 X C_2 e \rangle \langle B_1 Y D_1 f, B_2 Y D_2 f \rangle +$$

$$\begin{aligned}
& \dots + \langle A_1XC_1e, A_nXC_ne \rangle \langle B_1YD_1f, B_nYD_nf \rangle + \langle A_2XC_2e, A_1XC_1e \rangle \langle B_2YD_2f, B_1YD_1f \rangle + \\
& \langle A_2XC_2e, A_2XC_2e \rangle \langle B_2YD_2f, B_2YD_2f \rangle + \dots + \langle A_2XC_2e, A_nXC_ne \rangle \langle B_2YD_2f, B_nYD_nf \rangle + \\
& \dots + \langle A_nXC_ne, A_1XC_1e \rangle \langle B_nYD_nf, B_1YD_1f \rangle + \langle A_nXC_ne, A_2XC_2e \rangle \langle B_nYD_nf, B_2YD_2f \rangle + \\
& \dots + \langle A_nXC_ne, A_nXC_ne \rangle \langle B_nYD_nf, B_nYD_nf \rangle \\
& = \|A_1XC_1e\|^2 \|B_1YD_1f\|^2 + \langle A_1XC_1e, A_2XC_2e \rangle \langle B_1YD_1f, B_2YD_2f \rangle + \dots + \\
& \langle A_1XC_1e, A_nXC_ne \rangle \langle B_1YD_1f, B_nYD_nf \rangle + \langle A_2XC_2e, A_1XC_1e \rangle \langle B_2YD_2f, B_1YD_1f \rangle + \\
& \|A_2XC_2e\|^2 \|B_2YD_2f\|^2 + \dots + \langle A_2XC_2e, A_nXC_ne \rangle \langle B_2YD_2f, B_nYD_nf \rangle + \dots + \\
& \langle A_nXC_ne, A_1XC_1e \rangle \langle B_nYD_nf, B_1YD_1f \rangle + \langle A_nXC_ne, A_2XC_2e \rangle \langle B_nYD_nf, B_2YD_2f \rangle + \\
& \dots + \|A_nXC_ne\|^2 \|B_nYD_nf\|^2 \tag{4.2}
\end{aligned}$$

Now, let $u_i, v_i : H \rightarrow \mathbb{R}^+$ be functionals, for $i = 1, 2, \dots, n$.

Choosing unit vectors $y, z \in H$ and define the finite rank operators

$A_i = u_i \otimes y, y \in H, \|y\| = 1$ by $A_i e = (u_i \otimes y)e = u_i(e)y$ $i = 1, 2, \dots, n$ and

$C_i = v_i \otimes z, z \in H, \|z\| = 1$ by $C_i e = (v_i \otimes z)e = v_i(e)z$ $i = 1, 2, \dots, n$

We observe that the norm of A_i is

$$\begin{aligned}
\|A_i\| &= \sup\{\|(u_i \otimes y)e\| : e \in H, \|y\| = 1\} \\
&= \sup\{\|u_i(e)y\| : e \in H, \|y\| = 1\} \\
&= \sup\{|u_i(e)| \|y\| : e \in H, \|y\| = 1\} \\
&= \sup\{|u_i(e)| : e \in H\} = |u_i(e)|
\end{aligned}$$

That is $\|A_i\| = |u_i(e)|$ for any $e \in H$.

Likewise using the concept above, the norm of C_i is $\|C_i\| = |v_i(e)|$ for any $e \in H$ with $i = 1, 2, \dots, n$.

$$\begin{aligned}
\text{Therefore, From (i) we have } \|A_1XC_1e\|^2 &= \|(u_1 \otimes y)X(v_1 \otimes z)e\|^2 \\
&= \|(u_1 \otimes y)X(v_1 \otimes z)e\|^2 = \|(u_1 \otimes y)X(v_1(e)z)\|^2 \\
&= \|(u_1 \otimes y)X(v_1(e)z)\|^2 = |v_1(e)|^2 \|(u_1 \otimes y)X(z)\|^2 \\
&= |v_1(e)|^2 \|u_1(X(z))y\|^2 = |v_1(e)|^2 |u_1(X(z))|^2 \|y\|^2 = \|A_1\|^2 \|C_1\|^2.
\end{aligned}$$

Then this shows that :

$$\|A_1XC_1e\|^2 = \|A_1\|^2 \|C_1\|^2 \tag{4.3}$$

Thus using the same concept also:

$$\|B_1 Y D_1 f\|^2 = \|B_1\|^2 \|D_1\|^2 \quad (4.4)$$

$$\|A_2 X C_2 e\|^2 = \|A_2\|^2 \|C_2\|^2 \quad (4.5)$$

$$\|B_2 Y D_2 f\|^2 = \|B_2\|^2 \|D_2\|^2 \quad (4.6)$$

$$\|A_n X C_n e\|^2 = \|A_n\|^2 \|C_n\|^2 \quad (4.7)$$

$$\|B_n Y D_n f\|^2 = \|B_n\|^2 \|D_n\|^2 \quad (4.8)$$

$$\begin{aligned} \text{Also } \langle A_1 X C_1 e, A_2 X C_2 e \rangle &= \langle (u_1 \otimes y)X(v_1 \otimes z))e, (u_2 \otimes y)X(v_2 \otimes z))e \rangle \\ &= \langle (u_1 \otimes y)X(v_1(e)z), (u_2 \otimes y)X(v_2(e)z) \rangle \\ &= \langle v_1(e)(u_1 \otimes y)X(z), v_2(e)(u_2 \otimes y)X(z) \rangle \\ &= \langle v_1(e)u_1(X(z))y, v_2(e)u_2(X(z))y \rangle \\ &= v_1(e)u_1(X(z))v_2(e)u_2(X(z))\langle y, y \rangle \\ &= v_1(e)u_1(X(z))v_2(e)u_2(X(z)) \end{aligned}$$

Since $v_1(e), u_1(X(z)), v_2(e)$ and $u_2(X(z))$ are positive real numbers, we have:

$$v_1(e) = |v_1(e)| = \|C_1\|, u_1(X(z)) = |u_1(X(z))| = \|A_1\|, v_2(e) = |v_2(e)| = \|C_2\|$$

$$\text{and } u_2(X(z)) = |u_2(X(z))| = \|A_2\|$$

$$\text{Thus we have } \langle A_1 X C_1 e, A_2 X C_2 e \rangle = v_1(e)u_1(X(z))v_2(e)u_2(X(z)) = \|C_1\| \|A_1\| \|C_2\| \|A_2\|$$

Since the norms of A_i and C_i for $i = 1, 2, \dots, n$ are scalars then:

$$\langle A_1 X C_1 e, A_2 X C_2 e \rangle = \|A_1\| \|A_2\| \|C_1\| \|C_2\| \quad (4.9)$$

Hence using the same concept as above then:

$$\langle B_1 Y D_1 f, B_2 Y D_2 f \rangle = \|B_1\| \|B_2\| \|D_1\| \|D_2\| \quad (4.10)$$

It then follows that;

$$\begin{aligned} \langle A_2 X C_2 e, A_1 X C_1 e \rangle &= \langle (u_2 \otimes y)X(v_2 \otimes z))e, (u_1 \otimes y)X(v_1 \otimes z))e \rangle \\ &= \langle (u_2 \otimes y)X(v_2(e)z), (u_1 \otimes y)X(v_1(e)z) \rangle \\ &= \langle v_2(e)(u_2 \otimes y)X(z), v_1(e)(u_1 \otimes y)X(z) \rangle \\ &= \langle v_2(e)u_2(X(z))y, v_1(e)u_1(X(z))y \rangle \\ &= v_2(e)u_2(X(z))v_1(e)u_1(X(z))\langle y, y \rangle \end{aligned}$$

$$= v_2(e)u_2(X(z))v_1(e)u_1(X(z)) = \|C_2\|\|A_2\|\|C_1\|\|A_1\|$$

Since the norms of A_i and C_i for $i = 1, 2, \dots, n$ are scalars then:

$$\langle A_2XC_2e, A_1XC_1e \rangle = \|A_1\|\|A_2\|\|C_1\|\|C_2\| \quad (4.11)$$

Thus using the same concept then:

$$\langle B_2YD_2f, B_1YD_1f \rangle = \|B_1\|\|B_2\|\|D_1\|\|D_2\| \quad (4.12)$$

$$\langle B_1YD_1f, B_nYD_nf \rangle = \|B_1\|\|B_n\|\|D_1\|\|D_n\| \quad (4.13)$$

$$\langle B_2YD_2f, B_nYD_nf \rangle = \|B_2\|\|B_n\|\|D_2\|\|D_n\| \quad (4.14)$$

$$\langle B_nYD_nf, B_1YD_1f \rangle = \|B_1\|\|B_n\|\|D_1\|\|D_n\| \quad (4.15)$$

$$\langle B_nYD_nf, B_2YD_2f \rangle = \|B_2\|\|B_n\|\|D_2\|\|D_n\| \quad (4.16)$$

$$\langle A_1XC_1e, A_nXC_ne \rangle = \|A_1\|\|A_n\|\|C_1\|\|C_n\| \quad (4.17)$$

$$\langle A_2XC_2e, A_nXC_ne \rangle = \|A_2\|\|A_n\|\|C_2\|\|C_n\| \quad (4.18)$$

$$\langle A_nXC_ne, A_1XC_1e \rangle = \|A_1\|\|A_n\|\|C_1\|\|C_n\| \quad (4.19)$$

$$\langle A_nXC_ne, A_2XC_2e \rangle = \|A_2\|\|A_n\|\|C_2\|\|C_n\| \quad (4.20)$$

Thus substituting equations (4.3) to (4.20) in (4.2) then:

$$\begin{aligned} \|T_n\|^2 &\geq \|A_1\|^2\|B_1\|^2\|C_1\|^2\|D_1\|^2 + \|A_1\|\|B_1\|\|C_1\|\|D_1\|\|A_2\|\|B_2\|\|C_2\|\|D_2\| \\ &+ \dots + \|A_1\|\|B_1\|\|C_1\|\|D_1\|\|A_n\|\|B_n\|\|C_n\|\|D_n\| + \|A_1\|\|B_1\|\|C_1\|\|D_1\|\|A_2\|\|B_2\|\|C_2\|\|D_2\| \\ &+ \|A_2\|^2\|B_2\|^2\|C_2\|^2\|D_2\|^2 + \dots + \|A_2\|\|B_2\|\|C_2\|\|D_2\|\|A_n\|\|B_n\|\|C_n\|\|D_n\| + \dots \\ &+ \|A_1\|\|B_1\|\|C_1\|\|D_1\|\|A_n\|\|B_n\|\|C_n\|\|D_n\| + \|A_2\|\|B_2\|\|C_2\|\|D_2\|\|A_n\|\|B_n\|\|C_n\|\|D_n\| + \\ &\dots + \|A_n\|^2\|B_n\|^2\|C_n\|^2\|D_n\|^2 \end{aligned}$$

This implies that:

$$\begin{aligned} \|T_n\|^2 &\geq \{\|A_1\|\|B_1\|\|C_1\|\|D_1\|\}^2 + 2\|A_1\|\|B_1\|\|C_1\|\|D_1\|\|A_2\|\|B_2\|\|C_2\|\|D_2\| + \dots \\ &+ 2\|A_1\|\|B_1\|\|C_1\|\|D_1\|\|A_n\|\|B_n\|\|C_n\|\|D_n\| + \dots + \{\|A_2\|\|B_2\|\|C_2\|\|D_2\|\}^2 + \dots \\ &+ 2\|A_2\|\|B_2\|\|C_2\|\|D_2\|\|A_n\|\|B_n\|\|C_n\|\|D_n\| + \dots + \|A_n\|^2\|B_n\|^2\|C_n\|^2\|D_n\|^2 \end{aligned}$$

Thus:

$$\|T_n\|^2 \geq \{\|A_1\|\|B_1\|\|C_1\|\|D_1\| + \|A_2\|\|B_2\|\|C_2\|\|D_2\| + \dots + \|A_n\|\|B_n\|\|C_n\|\|D_n\|\}^2$$

Thus obtaining square root in both sides we get;

$$\|T_n\| \geq \|A_1\| \|B_1\| \|C_1\| \|D_1\| + \|A_2\| \|B_2\| \|C_2\| \|D_2\| + \dots + \|A_n\| \|B_n\| \|C_n\| \|D_n\|$$

Finally its clear that:

$$\|T_n\| \geq \sum_{i=1}^n \|A_i\| \|B_i\| \|C_i\| \|D_i\| \quad (4.21)$$

From (4.1) and (4.21) then ;

$$\|T_n \setminus B(H \otimes K)\| = \sum_{i=1}^n \|A_i\| \|B_i\| \|C_i\| \|D_i\| \quad (4.22)$$

□

4.3 Norm of Jordan Elementary Operator in Tensor Product of C^* -algebras

In this section, the study investigated the bounds of the norm of Jordan elementary operator in a tensor product of C^* -algebras. Finally, as consequence, the study shown that the norm of Jordan elementary operator in $B(H \otimes K)$ can be expressed in terms of norms of Jordan elementary operators on $B(H)$ and $B(K)$.

Theorem 4.3.1:

Let $H \otimes K$ be tensor product of Hilbert spaces H and K and $B(H \otimes K)$ be the set of bounded linear operators on $H \otimes K$. Then $\forall X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$ then $\|U_{A \otimes B, C \otimes D}\| = 2\|A\| \|B\| \|C\| \|D\|$, where $U_{A \otimes B, C \otimes D}$ is the Jordan elementary operator in a tensor product of C^* -algebras as defined earlier and A, C and B, D are in $B(H)$ and $B(K)$ respectively.

Proof. By definition;

$$\|U_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| = \text{Sup}\{\|U_{A \otimes B, C \otimes D}(X \otimes Y)\| : X \otimes Y \in B(H \otimes K), \|X \otimes Y\| = 1\}$$

For any $U_{A \otimes B, C \otimes D} : B(H \otimes K) \rightarrow B(H \otimes K)$ and $X \otimes Y : H \otimes K \rightarrow H \otimes K$ then,

$$\|U_{A \otimes B, C \otimes D}(X \otimes Y)\| \leq \|U_{A \otimes B, C \otimes D}\| \|X \otimes Y\|.$$

Or;

$$\|U_{A \otimes B, C \otimes D}\| \|X \otimes Y\| \geq \|U_{A \otimes B, C \otimes D}(X \otimes Y)\|.$$

By choosing $\varepsilon \in (0, \|U_{A \otimes B, C \otimes D}\|)$. Then $\|U_{A \otimes B, C \otimes D}\| - \varepsilon > 0$.

Since $\|U_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| = \text{Sup}\{\|U_{A \otimes B, C \otimes D}(X \otimes Y)\| : X \otimes Y \in B(H \otimes K), \|X \otimes Y\| = 1\}$.

Then $\exists X' \otimes Y' \in B(H \otimes K)$ with $\|X' \otimes Y'\| \neq 0$ for which;

$$(\|U_{A \otimes B, C \otimes D}\| - \varepsilon)\|X' \otimes Y'\| < \|U_{A \otimes B, C \otimes D}(X' \otimes Y')\|.$$

That is;

$$\|U_{A \otimes B, C \otimes D}\| - \varepsilon < \frac{1}{\|X' \otimes Y'\|} \|U_{A \otimes B, C \otimes D}(X' \otimes Y')\|.$$

This implies that;

$$\|U_{A \otimes B, C \otimes D}\| - \varepsilon < \|U_{A \otimes B, C \otimes D} \frac{X' \otimes Y'}{\|X' \otimes Y'\|}\|.$$

Since $\frac{X' \otimes Y'}{\|X' \otimes Y'\|}$ is a unit in $B(H \otimes K)$ then;

$$\|U_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| - \varepsilon < \|U_{A \otimes B, C \otimes D}(X \otimes Y)\| \text{ where } X \otimes Y \in B(H \otimes K) \text{ with } \|X \otimes Y\| = 1.$$

Since ε was arbitrarily chosen then;

$$\|U_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| \leq \|U_{A \otimes B, C \otimes D}(X \otimes Y)\| \text{ where } X \otimes Y \in B(H \otimes K) \text{ with } \|X \otimes Y\| = 1.$$

Because $\|U_{A \otimes B, C \otimes D}(X \otimes Y)\| = \|A \otimes B(X \otimes Y)C \otimes D + C \otimes D(X \otimes Y)A \otimes B\| : X \otimes Y \in B(H \otimes K), \|X \otimes Y\| = 1$ then;

$$\|U_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| \leq \|A \otimes B(X \otimes Y)C \otimes D + C \otimes D(X \otimes Y)A \otimes B\| : X \otimes Y \in B(H \otimes K), \|X \otimes Y\| = 1 \leq \|A \otimes B(X \otimes Y)C \otimes D\| + \|C \otimes D(X \otimes Y)A \otimes B\| : X \otimes Y \in B(H \otimes K), \|X \otimes Y\| = 1 \text{ (By triangular property)}.$$

Then;

$$\|U_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| \leq \|A \otimes B(X \otimes Y)C \otimes D\| + \|C \otimes D(X \otimes Y)A \otimes B\| : X \otimes Y \in$$

$$B(H \otimes K), \|X \otimes Y\| = 1.$$

Now using the properties of tensor products $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ then;

$$\|U_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| \leq \|AXC \otimes BYD\| + \|CXA \otimes DYB\|; \|X \otimes Y\| = 1.$$

By properties of tensor product of operators $\|A \otimes B\| = \|A\|\|B\|$ we have;

$$\|U_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| \leq \|AXC\|\|BYD\| + \|CXA\|\|DYB\|$$

$$; \|X \otimes Y\| = 1.$$

Clearly;

$$\|U_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| \leq \|A\|\|X\|\|C\|\|B\|\|Y\|\|D\| + \|C\|\|X\|\|A\|\|D\|\|Y\|\|B\|$$

$$; \|X \otimes Y\| = 1.$$

Clearly;

$$\|X \otimes Y\| = \|X\|\|Y\| = 1.$$

Then;

$$\|U_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| \leq \|A\|\|C\|\|B\|\|D\| + \|C\|\|A\|\|D\|\|B\|.$$

Finally then;

$$\|U_{A \otimes B, C \otimes D}\| \leq 2\|A\|\|B\|\|C\|\|D\|.$$

Conversely, for $U_{A \otimes B, C \otimes D} : B(H \otimes K) \rightarrow B(H \otimes K)$ and $X \otimes Y : H \otimes K \rightarrow H \otimes K$, let there exist unit vector $\mu \otimes \nu$ of $H \otimes K$ where $\mu \in H$ and $\nu \in K$

Then;

$$\|U_{A \otimes B, C \otimes D}(X \otimes Y)(\mu \otimes \nu)\| \leq \|U_{A \otimes B, C \otimes D}(X \otimes Y)\|\|\mu \otimes \nu\| \leq \|U_{A \otimes B, C \otimes D}\|\|X \otimes Y\|\|\mu \otimes \nu\|$$

$$= \|U_{A \otimes B, C \otimes D}\|\|X\|\|Y\|\|\mu\|\|\nu\|$$

$$\|X\| = \|Y\| = \|\mu\| = \|\nu\| = 1.$$

Thus

$$\|U_{A \otimes B, C \otimes D}\| \geq \|U_{A \otimes B, C \otimes D}(X \otimes Y)(\mu \otimes \nu)\| = \|(A \otimes B(X \otimes Y)C \otimes D + C \otimes D(X \otimes Y)A \otimes B)(\mu \otimes \nu)\|$$

By the properties of tensor product on operators $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ then;

$$= \|AXC \otimes BYD(\mu \otimes \nu) + CXA \otimes DYB(\mu \otimes \nu)\|$$

$$= \|AXC\mu \otimes BYD\nu + CXA\mu \otimes DYB\nu\|$$

Therefore

$$\|U_{A \otimes B, C \otimes D}\| \geq \|AXC\mu \otimes BYD\nu + CXA\mu \otimes DYB\nu\| \quad (4.23)$$

Squaring both sides of (4.23) we get

$$\begin{aligned} \{\|U_{A \otimes B, C \otimes D}\|\}^2 &\geq \{\|AXC\mu \otimes BYD\nu + CXA\mu \otimes DYB\nu\|\}^2 \\ &= \langle AXC\mu \otimes BYD\nu + CXA\mu \otimes DYB\nu, AXC\mu \otimes BYD\nu + CXA\mu \otimes DYB\nu \rangle \\ &= \langle AXC\mu \otimes BYD\nu, AXC\mu \otimes BYD\nu + CXA\mu \otimes DYB\nu \rangle + \langle CXA\mu \otimes \\ &\quad DYB\nu, AXC\mu \otimes BYD\nu + CXA\mu \otimes DYB\nu \rangle \\ &= \langle AXC\mu \otimes BYD\nu, AXC\mu \otimes BYD\nu \rangle + \langle AXC\mu \otimes BYD\nu, CXA\mu \otimes DYB\nu \rangle + \\ &\quad \langle CXA\mu \otimes DYB\nu, AXC\mu \otimes BYD\nu \rangle + \langle CXA\mu \otimes DYB\nu, CXA\mu \otimes DYB\nu \rangle \\ &= \|AXC\mu \otimes BYD\nu\|^2 + \langle AXC\mu \otimes BYD\nu, CXA\mu \otimes DYB\nu \rangle + \langle CXA\mu \otimes \\ &\quad DYB\nu, AXC\mu \otimes BYD\nu \rangle + \|CXA\mu \otimes DYB\nu\|^2 \\ &= \|AXC\mu\|^2 \|BYD\nu\|^2 + \langle AXC\mu \otimes BYD\nu, CXA\mu \otimes DYB\nu \rangle + \langle CXA\mu \otimes \\ &\quad DYB\nu, AXC\mu \otimes BYD\nu \rangle + \|CXA\mu\|^2 \|DYB\nu\|^2 \\ &= \|AXC\mu\|^2 \|BYD\nu\|^2 + \langle AXC\mu, CXA\mu \rangle \langle BYD\nu, DYB\nu \rangle + \langle CXA\mu, AXC\mu \rangle \langle DYB\nu, BYD\nu \rangle + \\ &\quad \|CXA\mu\|^2 \|DYB\nu\|^2 \end{aligned} \quad (4.24)$$

We now let $e, f : H \rightarrow \mathbb{R}^+$ be functionals

From above, Choosing unit vectors $y, z \in H$ and define the finite rank operators

$$A = e \otimes y, y \in H, \|y\| = 1 \text{ by } A\mu = (e \otimes y)\mu = e(\mu)y \text{ and}$$

$$C = f \otimes z, z \in H, \|z\| = 1 \text{ by } C\mu = (f \otimes z)\mu = f(\mu)z$$

We observe that the norm of A is

$$\begin{aligned} \|A\| &= \sup\{\|(e \otimes y)\mu\| : \mu \in H, \|y\| = 1\} \\ &= \sup\{\|e(\mu)y\| : \mu \in H, \|y\| = 1\} \\ &= \sup\{|e(\mu)| \|y\| : \mu \in H, \|y\| = 1\} \\ &= \sup\{|e(\mu)| : \mu \in H, \|\cdot\| = 1\} = |e(\mu)| \end{aligned}$$

That is $\|A\| = |e(\mu)|$ for any $\mu \in H$.

Likewise using the same concept, the norm of A is $\|C\| = |f(\mu)|$ for any $\mu \in H$

$$\begin{aligned}
\text{Therefore, we have } \|AXC\mu\|^2 &= \|(e \otimes y)X(f \otimes z)\mu\|^2 \\
&= \|(e \otimes y)X(f \otimes z)\mu\|^2 = \|(e \otimes y)X(f(\mu)z)\|^2 \\
&= \|(e \otimes y)X(f(\mu)z)\|^2 = |f(\mu)|^2 \|(e \otimes y)X(z)\|^2 \\
&= |f(\mu)|^2 \|e(X(z))y\|^2 = |f(\mu)|^2 |e(X(z))|^2 \|y\|^2 = \|A\|^2 \|C\|^2.
\end{aligned}$$

Then this shows that :

$$\|AXC\mu\|^2 = \|A\|^2 \|C\|^2 \quad (4.25)$$

Using the same concept we have;

$$\|BYD\nu\|^2 = \|B\|^2 \|D\|^2 \quad (4.26)$$

$$\|DYB\nu\|^2 = \|B\|^2 \|D\|^2 \quad (4.27)$$

$$\|CXA\mu\|^2 = \|A\|^2 \|C\|^2 \quad (4.28)$$

$$\begin{aligned}
\text{Also } \langle AXC\mu, CXA\mu \rangle &= \langle (e \otimes y)X(f \otimes z)\mu, (e \otimes y)X(f \otimes z)\mu \rangle \\
&= \langle (e \otimes y)X(f(\mu)z), (e \otimes y)X(f(\mu)z) \rangle \\
&= \langle f(\mu)(e \otimes y)X(z), f(\mu)(e \otimes y)X(z) \rangle \\
&= \langle f(\mu)e(X(z))y, f(\mu)e(X(z))y \rangle \\
&= f(\mu)e(X(z))f(\mu)e(X(z))\langle y, y \rangle \\
&= f(\mu)e(X(z))f(\mu)e(X(z))
\end{aligned}$$

Since $f(\mu)$ and $e(X(z))$ are positive real numbers, we have:

$$f(\mu) = |f(\mu)| = \|A\|, e(X(z)) = |e(X(z))| = \|C\|$$

$$\text{Thus we have } \langle AXC\mu, CXA\mu \rangle = f(\mu)e(X(z))f(\mu)e(X(z)) = \|C\| \|A\| \|C\| \|A\|$$

Since the norms of A and C are scalars then:

$$\langle AXC\mu, CXA\mu \rangle = \|A\|^2 \|C\|^2 \quad (4.29)$$

Then following the same concept we have;

$$\langle CXA\mu, AXC\mu \rangle = \|A\|^2 \|C\|^2 \quad (4.30)$$

$$\langle BYD\nu, DXB\nu \rangle = \|B\|^2 \|D\|^2 \quad (4.31)$$

$$\langle DYB\nu, BXD\nu \rangle = \|B\|^2\|D\|^2 \quad (4.32)$$

Then applying equations (4.25) to (4.32) to equation (4.24) we have

$$= \|A\|^2\|B\|^2\|C\|^2\|D\|^2 + \|A\|^2\|B\|^2\|C\|^2\|D\|^2 + \|A\|^2\|B\|^2\|C\|^2\|D\|^2 + \|A\|^2\|B\|^2\|C\|^2\|D\|^2$$

Thus

$$\{\|U_{A\otimes B, C\otimes D}\|\}^2 \geq \|A\|^2\|B\|^2\|C\|^2\|D\|^2 + \|A\|^2\|B\|^2\|C\|^2\|D\|^2 + \|A\|^2\|B\|^2\|C\|^2\|D\|^2 + \|A\|^2\|B\|^2\|C\|^2\|D\|^2$$

Then

$$\{\|U_{A\otimes B, C\otimes D}\|\}^2 \geq 4\|A\|^2\|B\|^2\|C\|^2\|D\|^2 \quad (4.33)$$

Obtaining square root on both sides of (4.33) we obtain;

$$\|U_{A\otimes B, C\otimes D}\| \geq 2\|A\|\|B\|\|C\|\|D\| \quad (4.34)$$

Hence we have;

$$\|U_{A\otimes B, C\otimes D}\| = 2\|A\|\|B\|\|C\|\|D\|$$

□

Corollary 4.3.2: Let $H \otimes K$ be tensor product of complex Hilbert spaces H and K and $B(H \otimes K)$ be the set of bounded linear operators on $H \otimes K$. Then $\forall X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$ then

$$\|U_{A\otimes B, C\otimes D}\| = 2\|U_{A, C}\|\|U_{B, D}\|$$

Proof. Blanco *et al.* (2004) determined the norm of Jordan elementary operator on a C^* -algebra using the notion of matrix-valued numerical ranges and kind of geometrical mean of positive matrices and tracial geometrical mean and obtained that;

$$\|AXC + CXA\| = \|A\|\|C\|, \forall A, C \in B(H)$$

Thus we can also conclude that;

$$\|BXD + DXB\| = \|B\|\|D\|, \forall B, D \in B(K)$$

So now using theorem 4.2.3

$$\|U_{A\otimes D, C\otimes B}\| = 2\|A\|\|B\|\|C\|\|D\|$$

By rearranging the above we get;

$$\|U_{A \otimes D, C \otimes D}\| = 2\|A\|\|C\|\|B\|\|D\|$$

Now using the work of Blanco *et al.*(2004) on the above we obtain;

$$\|U_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| = 2\|U_{A, C} \setminus B(H)\|\|U_{B, D} \setminus B(K)\|$$

□

4.4 Norm of an Arbitrary Elementary Operator in Tensor Product of C*-algebras Expressible in terms of the Sum of Product of Norms of Corresponding Coefficient Operators.

Finally, in this section, the definition of maximal numerical range for a tensor product of operators and the conditions necessary for the norm of finite length Elementary Operator in tensor product of C*-algebras to be equal to the sum of product of norms of corresponding coefficient operators in its definition were determined.

Definition 4.4.1: Maximal Numerical Range of Tensor Product of Operators

Let $A_i \otimes B_i \in B(H \otimes K)$. If $\|A_i \otimes B_i\| \in W_o(A_i \otimes B_i)$ then $\lim_{k \rightarrow \infty} \langle (A_i \otimes B_i)(x_k \otimes y_k, (x_k \otimes y_k)) \rangle = \|A_i\|\|B_i\| = \|A_i \otimes B_i\|$ and $\lim_{k \rightarrow \infty} \|(A_i \otimes B_i)(x_k \otimes y_k)\| = \|(A_i \otimes B_i)\| = \|A_i\|\|B_i\| \forall i = 1, 2, 3, \dots, n$.

This is true since for if we let $\{x_k \otimes y_k\}_{k \geq 1}$ be a sequence of unit vectors on $H \otimes K$ and $(x_k \otimes y_k) \otimes (x_k \otimes y_k) \in B(H \otimes K)$ be rank one operator on $H \otimes K$ for a unit vector $x_k \otimes y_k \in H \otimes K$ defined by $(x_k \otimes y_k) \otimes (x_k \otimes y_k)(x \otimes y) = \langle x \otimes y, x_k \otimes y_k \rangle x_k \otimes y_k, \forall x \otimes y \in H \otimes K$ then;

$\|A_i \otimes B_i\| \in w_o(A_i \otimes B_i)$ then there is a sequence $\{x_k \otimes y_k\}_{k \geq 1}$ of unit vectors on $H \otimes K$ such that

$$\begin{aligned} \lim_{k \rightarrow \infty} \langle (A_i \otimes B_i)(x_k \otimes y_k), (x_k \otimes y_k) \rangle &= \lim_{k \rightarrow \infty} \langle (A_i x_k \otimes B_i y_k), (x_k \otimes y_k) \rangle = \\ \lim_{k \rightarrow \infty} \langle A_i x_k, x_k \rangle \langle B_i y_k, y_k \rangle &= \lim_{k \rightarrow \infty} \langle A_i x_k, x_k \rangle \lim_{k \rightarrow \infty} \langle B_i y_k, y_k \rangle = \|A_i\|\|B_i\| = \\ &= \|A_i \otimes B_i\| \end{aligned}$$

Also;

$$\begin{aligned} \lim_{k \rightarrow \infty} \|(A_i \otimes B_i)(x_k \otimes y_k)\| &= \lim_{k \rightarrow \infty} \|(A_i x_k \otimes B_i y_k)\| = \lim_{k \rightarrow \infty} \|(A_i x_k)\| \|B_i y_k\| = \\ &= \lim_{k \rightarrow \infty} \|(A_i x_k)\| \lim_{k \rightarrow \infty} \|B_i y_k\| = \|A_i\| \|B_i\| = \|A_i \otimes B_i\| \quad \forall i = 1, 2, 3, \dots, n \end{aligned}$$

Theorem 4.4.2:

Let T_n be finite length elementary operator on $B(H \otimes K)$ then $\|T_n\| = \sum_{i=1}^n \|A_i\| \|B_i\| \|C_i\| \|D_i\|$ if $\|A_i \otimes B_i\| \in W_o(A_i \otimes B_i)$ and $\|C_i \otimes D_i\| \in W_o(C_i \otimes D_i)$ with $A_i, C_i \in B(H)$ and $B_i, D_i \in B(K) \quad \forall i = 1, 2, \dots, n$,

Proof. By definition 4.4.1, let $\{x_k \otimes y_k\}_{k \geq 1}$ be a sequence of unit vectors on $H \otimes K$ and $(x_k \otimes y_k) \otimes (x_k \otimes y_k) \in B(H \otimes K)$ be rank one operator on $H \otimes K$ for a unit vector $x_k \otimes y_k \in H \otimes K$ defined by $(x_k \otimes y_k) \otimes (x_k \otimes y_k)(x \otimes y) = \langle x \otimes y, x_k \otimes y_k \rangle x_k \otimes y_k$, for all $x \otimes y \in H \otimes K$.

If $\|A_i \otimes B_i\| \in w_o(A_i \otimes B_i)$ and $\|C_i \otimes D_i\| \in w_o(C_i \otimes D_i)$ for $1, 2, 3, \dots, n$ then there is a sequence $\{x_k \otimes y_k\}_{k \geq 1}$ of unit vectors on $H \otimes K$ such that

$$\lim_{k \rightarrow \infty} \langle (A_i \otimes B_i)(x_k \otimes y_k), (x_k \otimes y_k) \rangle = \|A_i \otimes B_i\|$$

Also

$$\lim_{k \rightarrow \infty} \|(A_i \otimes B_i)(x_k \otimes y_k)\| = \|(A_i \otimes B_i)\| = \|A_i\| \|B_i\| \quad \forall i = 1, 2, \dots, n$$

Using the same concept, there is a sequence $\{x_k \otimes y_k\}_{k \geq 1}$ of unit vectors on $H \otimes K$ such that;

$$\lim_{k \rightarrow \infty} \langle (C_i \otimes D_i)(x_k \otimes y_k), x_k \otimes y_k \rangle = C_i \otimes D_i$$

Such that

$$\lim_{k \rightarrow \infty} \|(C_i \otimes D_i)x_k \otimes y_k\| = \|C_i \otimes D_i\| = \|C_i\| \|D_i\| \quad \forall 1, 2, 3, \dots, n.$$

For each $k \geq 1$, we have,

$$\begin{aligned} &\|(T_n((x_k \otimes y_k) \otimes (x_k \otimes y_k)))\| = \|T_{A_i \otimes B_i, C_i \otimes D_i}((x_k \otimes y_k) \otimes (x_k \otimes y_k))\| \\ &= \|T_{A_i \otimes B_i, C_i \otimes D_i}((x_k \otimes y_k) \otimes (x_k \otimes y_k)(x \otimes y))\| \leq \|T_{A_i \otimes B_i, C_i \otimes D_i}((x_k \otimes y_k) \otimes (x_k \otimes y_k))\| \|x \otimes y\| = \\ &\|T_{A_i \otimes B_i, C_i \otimes D_i}((x_k \otimes y_k))\| \|x_k \otimes y_k\| \|x \otimes y\| = \\ &\|T_{A_i \otimes B_i, C_i \otimes D_i}\| \|x_k\| \|y_k\| \|x_k\| \|y_k\| \|x\| \|y\| = \|T_{A_i \otimes B_i, C_i \otimes D_i}\| \end{aligned}$$

Since x_k, y_k, x, y are unit vectors

Thus we have,

$$\|T_{A_i \otimes B_i, C_i \otimes D_i}\| \geq \|T_{A_i \otimes B_i, C_i \otimes D_i}((x_k \otimes y_k) \otimes (x_k \otimes y_k)(x \otimes y))\|$$

Squaring both sides, we have

$$\{\|T_{A_i \otimes B_i, C_i \otimes D_i}\|\}^2 \geq \{\|T_{A_i \otimes B_i, C_i \otimes D_i}((x_k \otimes y_k) \otimes (x_k \otimes y_k)(x \otimes y))\|\}^2$$

By definition we have;

$$\begin{aligned} &= \|(A_1 \otimes B_1((x_k \otimes y_k) \otimes (x_k \otimes y_k))C_1 \otimes D_1 + A_2 \otimes B_2((x_k \otimes y_k) \otimes (x_k \otimes y_k))C_2 \otimes D_2 + \\ &\cdots + A_n \otimes B_n((x_k \otimes y_k) \otimes (x_k \otimes y_k))C_n \otimes D_n)(x \otimes y)\|^2 \\ &= \|(A_1 \otimes B_1((x_k \otimes y_k) \otimes (x_k \otimes y_k))C_1 \otimes D_1)(x \otimes y) + (A_2 \otimes B_2((x_k \otimes y_k) \otimes (x_k \otimes y_k))C_2 \otimes \\ &D_2)(x \otimes y) + \cdots + (A_n \otimes B_n((x_k \otimes y_k) \otimes (x_k \otimes y_k))C_n \otimes D_n)(x \otimes y)\|^2 \\ &= \|\langle (C_1 \otimes D_1)(x \otimes y), (x_k \otimes y_k) \rangle (A_1 \otimes B_1)(x_k \otimes y_k) + \langle (C_2 \otimes D_2)(x \otimes y), (x_k \otimes y_k) \rangle (A_2 \otimes \\ &B_2)(x_k \otimes y_k) + \cdots + \langle (C_n \otimes D_n)(x \otimes y), (x_k \otimes y_k) \rangle (A_n \otimes B_n)(x_k \otimes y_k)\|^2 \\ &= \langle \langle (C_1 \otimes D_1)(x \otimes y), (x_k \otimes y_k) \rangle (A_1 \otimes B_1)(x_k \otimes y_k) + \langle (C_2 \otimes D_2)(x \otimes y), (x_k \otimes y_k) \rangle (A_2 \otimes \\ &B_2)(x_k \otimes y_k) + \cdots + \langle (C_n \otimes D_n)(x \otimes y), (x_k \otimes y_k) \rangle (A_n \otimes B_n)(x_k \otimes y_k), \langle (C_1 \otimes D_1)(x \otimes \\ &y), (x_k \otimes y_k) \rangle (A_1 \otimes B_1)(x_k \otimes y_k) + \langle (C_2 \otimes D_2)(x \otimes y), (x_k \otimes y_k) \rangle (A_2 \otimes B_2)(x_k \otimes y_k) + \\ &\cdots + \langle (C_n \otimes D_n)(x \otimes y), (x_k \otimes y_k) \rangle (A_n \otimes B_n)(x_k \otimes y_k) \rangle \\ &= \|\langle C_1 \otimes D_1(x \otimes y), x_k \otimes y_k \rangle A_1 \otimes B_1(x_k \otimes y_k)\|^2 + \|\langle C_2 \otimes D_2(x \otimes y), x_k \otimes y_k \rangle A_2 \otimes B_2(x_k \otimes y_k)\|^2 + \\ &\cdots + \|\langle C_n \otimes D_n(x \otimes y), x_k \otimes y_k \rangle A_n \otimes B_n(x_k \otimes y_k)\|^2 + 2\text{Re}\langle \langle C_n \otimes D_n(x \otimes y), x_k \otimes y_k \rangle A_n \otimes \\ &B_n(x_k \otimes y_k), \dots, \langle C_3 \otimes D_3(x \otimes y), x_k \otimes y_k \rangle A_3 \otimes B_3(x_k \otimes y_k), \langle C_2 \otimes D_2(x \otimes y), x_k \otimes y_k \rangle A_2 \otimes B_2(x_k \otimes \\ &y_k), \langle C_1 \otimes D_1(x \otimes y), x_k \otimes y_k \rangle A_n \otimes B_n(x_k \otimes y_k) \rangle \\ &= \sum_{i=1}^n \sum_{k=1}^m \|\langle (C_i \otimes D_i)(x \otimes y), (x_k \otimes y_k) \rangle (A_i \otimes B_i)(x_k \otimes y_k)\|^2 \\ &\geq \sum_{i=1}^n \sum_{k=1}^m |\langle (C_i \otimes D_i)(x \otimes y), (x_k \otimes y_k) \rangle|^2 \|(A_i \otimes B_i)(x_k \otimes y_k)\|^2 \end{aligned}$$

Taking limits as $k \rightarrow \infty$, we have;

$$\geq \sum_{i=1}^n \|C_i \otimes D_i\|^2 \|A_i \otimes B_i\|^2 = \sum_{i=1}^n \|C_i\|^2 \|D_i\|^2 \|A_i\|^2 \|B_i\|^2$$

Thus

$$\|T_n\|^2 \geq \sum_{i=1}^n \|A_i\|^2 \|B_i\|^2 \|C_i\|^2 \|D_i\|^2$$

Hence

$$\|T_n\| \geq \sum_{i=1}^n \|A_i\| \|B_i\| \|C_i\| \|D_i\|$$

And since

$$\|T_n\| \leq \sum_{i=1}^n \|A_i\| \|B_i\| \|C_i\| \|D_i\|$$

holds by the triangular inequality we have

$$\|T_n\| = \sum_{i=1}^n \|A_i\| \|B_i\| \|C_i\| \|D_i\|$$

□

CHAPTER FIVE
CONCLUSION AND RECOMMENDATION

5.1 Conclusion

In section 4.1, the study extended the definition of elementary operator in tensor product of C^* -algebras studied by Muiruri *et al.* (2018) to other types of elementary operators like left multiplication operator, right multiplication operator, and general derivative operator. The study obtained that;

(a) *The left multiplication operator, $L_{A \otimes B} : B(H \otimes K) \rightarrow B(H \otimes K)$, is defined by;*

$$L_{A \otimes B}(X \otimes Y) = A \otimes B(X \otimes Y); \forall X \otimes Y \in B(H \otimes K),$$

where $A \otimes B$, being fixed element of $B(H \otimes K)$ with $A \in B(H)$, the set of bounded linear operators on H and $B \in B(K)$, the set of bounded linear operators in K .

(b) *The right multiplication operator, $R_{C \otimes D} : B(H \otimes K) \rightarrow B(H \otimes K)$, is defined by;*

$$R_{C \otimes D}(X \otimes Y) = C \otimes D(X \otimes Y); \forall X \otimes Y \in B(H \otimes K),$$

where $C \otimes D$, being fixed element of $B(H \otimes K)$ with $C \in B(H)$, the set of bounded linear operators on H and $D \in B(K)$, the set of bounded linear operators in K .

(c) *The general derivation operator is defined by;*

$$\delta_{A \otimes B, C \otimes D} = (A \otimes B)(X \otimes Y) - (X \otimes Y)(C \otimes D) = L_{A \otimes B} - R_{C \otimes D},$$

where $L_{A \otimes B}, R_{C \otimes D}$ are left and right multiplication operators as defined above.

(d) *The inner derivation operator is defined by;*

$$\delta_{A \otimes B, A \otimes B} = (A \otimes B)(X \otimes Y) - (X \otimes Y)(A \otimes B) = L_{A \otimes B} - R_{A \otimes B}.$$

For the section 4.2, the study's investigated the bounds of the norm of an elementary operator of finite length in a tensor product of C^* -algebras theorem 4.2.1. and obtained that;

$$\forall X \otimes Y \in B(H \otimes K) \text{ with } \|X \otimes Y\| = 1 \text{ then } \|T_n\| = \sum_{i=1}^n \|A_i\| \|B_i\| \|C_i\| \|D_i\|,$$

where T_n is the Elementary operator of finite length in a tensor product of C*-algebras and A_i, C_i and B_i, D_i are in $B(H)$ and $B(K)$ respectively.

In section 4.3, the study investigated the bounds of the norm of Jordan elementary operator in a tensor product of C*-algebras theorem 4.3.1 and obtained that;

$$\forall X \otimes Y \in B(H \otimes K) \text{ with } \|X \otimes Y\| = 1 \text{ then } \|U_{A \otimes B, C \otimes D}\| = 2\|A\| \|B\| \|C\| \|D\|,$$

where T_n is the Elementary operator of finite length in a tensor product of C*-algebras and A_i, C_i and B_i, D_i are in $B(H)$ and $B(K)$ respectively. Which was the study's specific objective two.

Also as a consequence of theorem 4.3.1, the study showed that the norm of Jordan elementary operator on $B(H \otimes K)$ can be expressed in terms of norms of Jordan elementary operators on $B(H)$ and $B(K)$ corollary 4.3.2 which is

$$\|U_{A \otimes B, C \otimes D} \setminus B(H \otimes K)\| = 2\|U_{A, C} \setminus B(H)\| \|U_{B, D} \setminus B(K)\|.$$

Finally, in section 4.4, the study defined the maximal numerical range in tensor product of operators and obtained that for $A_i \otimes B_i \in B(H \otimes K)$ such that $\|A_i \otimes B_i\| \in W_o(A_i \otimes B_i)$, then

$$\lim_{k \rightarrow \infty} \langle (A_i \otimes B_i)(x_k \otimes y_k, (x_k \otimes y_i)) \rangle = \|A_i \otimes B_i\|$$

and

$$\lim_{k \rightarrow \infty} \|(A_i \otimes B_i)(x_k \otimes y_k)\| = \|(A_i \otimes B_i)\| = \|A_i\| \|B_i\| \forall i = 1, 2, 3, \dots, n.$$

The conditions necessary for the norm of finite length Elementary Operator in tensor product of C*-algebras is equal to the sum of product of norms of corresponding coefficient operators in its definition were obtained and found that for a finite length elementary operator, T_n , on $B(H \otimes K)$ then $\|T_n\| = \sum_{i=1}^n \|A_i\| \|B_i\| \|C_i\| \|D_i\|$ if $\|A_i \otimes B_i\| \in W_o(A_i \otimes B_i)$ and $\|C_i \otimes D_i\| \in W_o(C_i \otimes D_i)$ with $A_i, C_i \in B(H)$ and $B_i, D_i \in B(K) \forall i = 1, 2, \dots, n$,

5.2 Suggestion for Further Study

In light of the research findings, we recommend to extend the results obtained in this thesis by;

- (a) In section 4.2, the study investigated the bounds of the norm of an elementary operator of finite length in a tensor product of C^* -algebras theorem 4.2.1 using the concept of finite rank operator, inner product and properties of tensor product the same can be investigated using numerical range or Stamfli's maximal numerical range.
- (b) For section 4.3, the study investigated the bounds of the norm of Jordan elementary operator in a tensor product of C^* -algebras theorem 4.3.1 using the concept of finite rank operator, inner product and properties of tensor product the same can be investigated using numerical range or Stamfli's maximal numerical range.
- (c) The norms of other types of elementary operators in tensor of C^* -algebras like left multiplication operator, right multiplication operator, general derivative operator, and inner derivation operator can also be studied using different applicable methods

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Postgraduate Studies Research Clearance (CHUKA UNIVERSITY)



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15th February, 2024

Director
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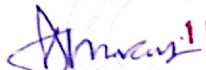
Dear Sir / Madam,

Peter Guchu Muiruri

The above-named person is a *bona fide* student of Chuka University pursuing PhD in Pure Mathematics proposal titled: **Norm of Finite Length Elementary Operator and Jordan Elementary Operator in Tensor Product of C^* -Algebras**

Mr. Muiruri has defended at the Faculty level and is now expected to conduct research. Any assistance accorded will be highly appreciated.

Yours sincerely,


15 FEB 2024
Prof. Moses Muraya, Ph.D.

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Appendix B
Research Authorization (CHUKA UNIVERSITY)



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6th February, 2024

REF: CUIERC/ NACOSTI/455

TO: Peter Guchu Muiruri

RE: Norm of Finite Length Elementary operator and Jordan Elementary Operator in Tensor Product of C^* Algebras

This is to inform you that *Chuka University IERC* has reviewed and approved your above research proposal. Your application approval number is *NACOSTI/NBC/AC-0812*. The approval period is 6th February, 2024 – 6th February, 2025.

This approval is subject to compliance with the following requirements;

- i. Only approved documents including (informed consents, study instruments, MTA) will be used
- ii. All changes including (amendments, deviations, and violations) are submitted for review and approval by *Chuka University IERC*.
- iii. Death and life threatening problems and serious adverse events or unexpected adverse events whether related or unrelated to the study must be reported to *Chuka University IERC* within 72 hours of notification
- iv. Any changes, anticipated or otherwise that may increase the risks or affected safety or welfare of study participants and others or affect the integrity of the research must be reported to *Chuka University IERC* within 72 hours
- v. Clearance for export of biological specimens must be obtained from relevant institutions.
- vi. Submission of a request for renewal of approval at least 60 days prior to expiry of the approval period. Attach a comprehensive progress report to support the renewal.
- vii. Submission of an executive summary report within 90 days upon completion of the study to *Chuka University IERC*.

Prior to commencing your study, you will be expected to obtain a research license from National Commission for Science, Technology and Innovation (NACOSTI) <https://oris.nacosti.go.ke> and also obtain other clearances needed.

Yours sincerely



Dr. Benjamin Kanga
SECRETARY

Appendix C

Research Permit (NACOSTI)



REPUBLIC OF KENYA

Ref No: **686190**

RESEARCH LICENSE



This is to Certify that Mr. **PETER GUCHU MUIRURI** of **Chuka University**, has been licensed to conduct research as per the provision of the Science, Technology and Innovation Act, 2013 (Rev.2014) in **Tharaka-Nithi** on the topic: **NORM OF FINITE LENGTH ELEMENTARY OPERATOR AND JORDAN ELEMENTARY OPERATOR IN TENSOR PRODUCT OF C*-ALGEBRAS** for the period ending : **24/February/2025**.

License No: **NACOSTI/P/24/33332**

686190

Applicant Identification Number

Director General
NATIONAL COMMISSION FOR
SCIENCE, TECHNOLOGY &
INNOVATION

Verification QR Code



NOTE: This is a computer generated License. To verify the authenticity of this document, Scan the QR Code using QR scanner application.

See overleaf for conditions

Appendix D

Publications

- (1) Muiruri, P., Kingangi, D., and Musundi, W., (2024). *On the Norm of finite length Elementary Operator in a Tensor Product of C^* -algebras*. International Journal of Mathematics and Computer Research,12(01):3991-3995.
- (2) Muiruri, P., Kingangi, D., and Musundi, W., (2024). *On the Norm of Jordan Elementary Operator in a Tensor Product of C^* -algebras*. GPH-International Journal of Mathematics, 7(03):01-10
- (3) Muiruri, P., Kingangi, D., and Musundi, W., (2024). *Conditions under which the Norm of finite length Elementary Operator in a Tensor Product of C^* -algebras is expansible in terms of the norm of its arbitrary operators*. (Under review)