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An Application Of Maximal Numerical Range On Norm Of Basic Elementary Operator In Tensor Product

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Abstract

Abstract

Many researchers in operator theory have attempted to determine the relationship between the norm of basic elementary operator and the norms of its coefficient operators. Various results have been obtained using varied approaches. In this paper, we attempt this problem by the use of the Stampfli's maximal numerical range in a tensor product.

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1.0 Introduction

1.1 Tensor products of Hilbert spaces

Definition 1.1.1. Tensor product. (Muiruri *et al,* 2019)

If $Z = \{u_{1,}u_2 \ldots\}$ and $L = \{v_1, v_2 \ldots\}$ are complex Hilbert spaces. Define their inner products $< u_1 u_2 >$ and $< v_2 v_2 >$ respectively. A tensor product of Z and L is a Hilbert space $Z \otimes L$ where $\otimes : Z \times L \to Z \otimes L, \otimes$ $(u, v) \to u \otimes v$ is a bilinear mapping:

i). The vectors $u \otimes v$ form a total subset of $Z \otimes L$

ii). $< u_1 \otimes v_1,\ u_2 \otimes v_2 > = < u_1, v_1 >< u_2, v_2 >, \forall\ u_1,\ u_2 \in Z, v_1, v_2 \in L$. This implies that $\| u \otimes v \| = \| u \| \| v \|\ \ \forall\ u \in Z, v \in L$.If $E \in B(Z), F \in B(L)$, then $B(Z \otimes L)$ is a Hilbert space and for $E \otimes F \in B(Z \otimes L)$ we have $E \otimes F(u \otimes v) = Eu \otimes Fv\ \forall\ u \in Z, v \in L$.

The following properties of members of $B(Z \otimes L)$ hold:

i). $(E \otimes F)(G \otimes H) = EG \otimes FH, \forall\ E \in B(Z), G \in B(Z)$ and $F \in B(L), H \in B(L)$.This property is both associative and commutative.

ii). $\| E \otimes F \| = \| E \| \| F \|\ \ \forall\ E \in B(Z)$ and $F \in B(L)$. This property indicates that norm is distributive under tensor product.

The linearity of the map, $\otimes\ (u, v) \to u \otimes v$ shows that $\otimes$ is linear with respect to the two coordinates, that is

(i) $(u_1 + u_2) \otimes v = (u_1 \otimes v) + (u_2 \otimes v)$

(ii) $(\psi u) \otimes v = \psi(u \otimes v)$.

(iii) $u \otimes (v_1 + v_2) = u \otimes v_1 + u \otimes v_2$

(iv) $u \otimes (\psi v) = \psi(u \otimes v)$.

(v) The set of all vectors $\otimes\ (u, v), u \epsilon Z$ and $v \epsilon L$ form a total subset of $Z \otimes L$.

Definition 1.1.2. Elementary operator in a tensor product. (Muiruri *et al*, 2019)

Let Z be a complex Hilbert space and L be complex Hilbert space, $B(Z \otimes L)$ be the collection of all bounded operators that are linear on the complex Hilbert space $Z \otimes L$ and $E \otimes F, G \otimes H$ be fixed elements of, $B(Z \otimes L)$ where $E, G \epsilon B(Z)$ and $F, H \epsilon B(K)$, the collection of bounded operators which are linear on Z and L respectively. Define the elementary operator as;

$$E_n(Z \otimes L) = \sum_{i=1}^{n} (E_i \otimes F_i)(U \otimes V)(G_i \otimes H_i) \qquad 1.1.1$$

for every $U \otimes V \epsilon B(Z \otimes L)$, $E_i \otimes F_i, G_i \otimes H_i$ being fixed elements of $B(Z \otimes L)$.

Now substituting for $n = 1$ in (1.1.1) we obtain the basic elementary operator,

$$E(Z \otimes L) = (E \otimes F)(U \otimes V)(G \otimes H) \qquad 1.1.2$$

From equation (1.1.2) the basic elementary operator can be expressed as,

$$E(Z \otimes L) = (E \otimes F)(U \otimes V)(G \otimes H) = (EUG) \otimes (FVH)$$

Definition 1.1.3. Stampfli's maximal numerical range of an operator. (Stampfli, 1970)

The Stampfli's maximal numerical range of an operator $G \in \mathcal{B}(Z)$ is the set

$$W_\circ(G) = \{\zeta \epsilon \mathbb{C} : < Gg_n, g_n > \to \zeta, \| g_n \| = 1, \| Gg_n \| \to \| g_n \|\}.$$

and Stampfli's maximal numerical range of an operator $G \in \mathcal{B}(L)$ is the set

$$W_{\circ}(H) = \{\xi \epsilon \mathbb{C}: < H\boxed{?}_n, \boxed{?}_n > \rightarrow \xi, \| \boxed{?}_n \| = 1, \| H\boxed{?}_n \| \rightarrow \| \boxed{?}_n \|\}.$$

2.0 Norm of Basic Elementary Operator

The norm of elementary operators has been investigated in the resent past under varied aspects. Their norms have been a subject of interest for research in operator theory. Deriving a formula to express the norm of an arbitrary elementary operator in terms of its coefficient operators remains a topic of research in operator theory. In the current paper, the concept of the maximal numerical range is applied in determining the lower bound of the norm of basic elementary operator. In particular the Stampfli's maximal numerical range is employed in arriving at our results. Okello (2011) utilized Dvoretsky theorem and its application in determining the norm of a symmetrized two-sided multiplication operator on a C^* algebra $B(H)$ and the result are as in lemma 2.1 and theorem 2.2;

Lemma 2.1: (Okelo, 2011)

Let H be a Hilbert space, $B(H)$ the algebra of bounded linear operators on H and a norm-attainable basic elementary operator $M_{A,B} : B\ (H) \rightarrow B(H)$ defined by $M_{A,B}(X) = AXB\ \forall\ X \in B(H)$ where $A,\ B$ are norm-attainable operators fixed in $B(H)$, then $||M_{A,B}|| = ||A||||B||$.

Theorem 2.2: (Okelo, 2011)

If $x, y \in B(H)$and let $x \otimes y$ denote the tensor product of x and y then

$$||x \otimes y + y \otimes x|| \leq \sqrt{2||x||^2||y||^2 + 2||y^*x||^2}$$

Okelo and Agure, (2011) also used the finite rank operator to determine the norm of the basic elementary operator and proved lemma 2.3;

Lemma 2.3: (Okelo and Agure, 2011)

Let H be a Hilbert space, $B(H)$ be the algebra of bounded linear operators on H .If $M_{A,B} : B\ (H) \rightarrow B(H)$ is defined by $M_{A,B}(X) = AXB$ where A, B are fixed elements in $B(H)$then

$$||M_{A,B}|| = ||A||||B||.$$

Boumazgour and Baraa (2008) used certain norm inequalities for 2×2 operator matrices to determine norm inequalities for sum of two basic elementary operators on a Hilbert space and obtained the norm inequality for sum of two basic elementary operator and proved the result as shown in theorem 2.4;

Theorem 2.4: (Boumazgour, 2008)

If A, B, C and D are operators in $B(H)$ then,

$$||M_{A,B} + M_{C,D}|| \leq [(max\{\ ||B||^2, ||D||^2\} + ||BD * ||)(max\{\ ||A||^2, ||C||^2\} + ||C * A||)]^{\frac{1}{2}}.$$

3.0 Main Results

3.1 Norm of Basic Elementary Operator in a Tensor Product

In this section the lower and upper bound of basic elementary operator is determined using the Stampfli's Maximal numerical range.

Lemma 3.2

For every $U \otimes V \epsilon \mathcal{B}(Z \otimes L), E_i \otimes F_i, G_i \otimes H_i$ being fixed elements of $\mathcal{B}(Z \otimes L)$ then, $(E \otimes F)(U \otimes VG \otimes H=(EUG) \otimes (FVH)$

Proof

By definition $(E \otimes F)(U \otimes V)(G \otimes H) = (EU \otimes FV)(G \otimes H)$

$$= (EUG) \otimes (FVH)$$

From lemma 3.2 $\forall\, E, G \,\epsilon\, B(Z)$ and $F, H \,\epsilon\, B(L)$ the basic elementary operator can also be defined as $O(H \otimes K=E \otimes FU \otimes VG \otimes H=(EUG) \otimes (FVH)$

The elementary operator of length two (size two) is obtained when $n = 2$;

$\forall\, U \otimes V \in \mathcal{B}(Z \otimes L), E_i \otimes F_i G_i \otimes H_i$ are fixed elements of $\mathcal{B}(Z \otimes L)$for $i = 1,2$

$$O_2(Z \otimes L) = \sum_{i=1}^{2} (E_i \otimes F_i)(U \otimes V)(G_i \otimes H_i)$$

$$= (E_1 \otimes F_1)(U \otimes V)(G_1 \otimes H_1) + (E_2 \otimes F_2)(U \otimes V)(G_2 \otimes H_2) \tag{3.2.1}$$

By lemma 3.2 equation (3) is expressed as follows,

$$O_2(Z \otimes L) = (E_1 \otimes F_1)(U \otimes V)(G_1 \otimes H_1) + (E_2 \otimes F_2)(U \otimes V)(G_2 \otimes H_2)$$

$$= (E_1 U F_1) \otimes (G_1 V H_1) + (E_2 U F_2) \otimes (G_2 V H_2)$$

By Stampfli's maximal numerical range, we determine the norm of basic elementary operator;

$$O(Z \otimes L) = O_{E \otimes F, G \otimes H}$$

$$= (E \otimes F)(U \otimes V)(G \otimes H)$$

$$= (EUF) \otimes (GVH).$$ []

Theorem 3.3

Let Z and L be complex Hilbert spaces and let, $O_{E \otimes F, G \otimes H}$ be the basic elementary operator on $\mathcal{B}(Z \otimes L)$ the set of bounded operators which are linear on a complex Hilbert space $Z \otimes L$. If $\forall\, U \otimes V \in \mathcal{B}(Z \otimes L)$ with $\|U \otimes V\| = 1, E, G \in B(Z), F, H \in B(L), \zeta \in W_{\circ}(G), \xi \in W_{\circ}(H)$ then we have,

$\|O_{E\otimes F,G\otimes H} \setminus B(Z \otimes L)\| = \mathrm{Sup}_{\zeta\in W_\circ(G)} \mathrm{Sup}_{\xi\in W_\circ(H)}\{|\zeta||\xi|\|E\|\|F\|\}$.

Proof.

By definition of the norm, $\forall\ U \otimes V \in \mathcal{B}(Z \otimes L), U \in (B(Z), V \in B(L), \|U\| = 1, \|V\| = 1$ then we have,

$\|O_{E\otimes F,G\otimes H} \setminus B(Z \otimes L)\| = Sup\|M_{E\otimes F,G\otimes H}\ (U \otimes V)\|$.

This implies that for every rank one operator $(m \otimes g_n)g_n = (g_n, g_n\rangle m \in B(H)$ and $(f \otimes h_n)h_n = (h_n, h_n\rangle f$ then,

$$\|O_{E\otimes F,G\otimes H} \setminus B(Z \otimes L)\| \geq \|O_{E\otimes F,G\otimes H}\ (m \otimes g_n)(g_n) \otimes (f \otimes h_n)(h_n)\|$$

$$\geq \|E \otimes F(m \otimes g_n)g_n \otimes (f \otimes h_n)h_n G \otimes H\|$$

$$\geq \|\{E(m \otimes g_n)g_n G\} \otimes \{F(f \otimes h_n)h_n H\|$$

$$\geq \|\{E(m \otimes g_n)Gg_n\} \otimes \{F(f \otimes h_n)Hh_n\}\|$$

$$\geq \|(Gg_n,g_n\rangle Em \otimes (Hh_n, h_n\rangle Ff\| \qquad (3.3.1)$$

By the definition 1.1.3, if $\zeta \in W_\circ(G)$ we have,

(i) $\lim_{n\to\infty}(Gg_n, g_n\rangle = \zeta$ and
(ii) $\lim_{n\to\infty}\|Gg_n\| = \|G\|$
Proof (ii)

$$\lim_{n\to\infty}\|Gg_n\| = \|G\|$$

$$= \|G \lim_{n\to\infty} g_n\|$$

$$= \|G\|$$

Since as $n \to \infty$ then $g_n \to 1$

and $\xi \in W_\circ(H)$ we have,

(i) $\lim_{n\to\infty}(Hh_n, h_n\rangle = \xi$ and
(ii) $\lim_{n\to\infty}\|Hh_n\| = \|H\|$
Proof (ii)

$$\lim_{n\to\infty}\|Hh_n\| = \|H\|$$

$$= \|Hh_n\|$$

$$= \|H\|$$

Since as $n \to \infty$ then $h_n \to 1$

Now taking limits as $n \to \infty$ on both sides of inequality (4)we have

$$\|O_{E\otimes F,G\otimes H} \backslash B(Z \otimes L)\| \geq \|(Gg_n, g_n\rangle Em \otimes (Hh_n, h_n\rangle Ff\|$$

$$\geq \|\zeta Em \otimes \xi Ff\|$$

$$\geq \|(\zeta \otimes \xi)(E \otimes F)(m \otimes f)\|$$

So $\forall \in > 0$,

$$\|O_{E\otimes F,G\otimes H} \backslash B(Z \otimes L)\| - \in < \|(\zeta \otimes \xi)(E \otimes F)(m \otimes f)\|$$

$$\leq \|(\zeta \otimes \xi)\| \|(E \otimes F)\| \|(m \otimes f)\|$$

$$\leq |\zeta||\xi|\|E\|\|F\|\|m\|\|f\|$$

$$\leq |\zeta||\xi|\|E\|\|F\| \text{ since } \|m\| = 1 \ and\ \|f\| = 1$$

where m and f are unit vectors in $B(Z)$ and $B(L)$ respectively.

Now since ϵ is arbitrary chosen and the unit vectors are chosen arbitrary then we get the supremum, this implies that

$$\|O_{E\otimes F,G\otimes H}\| \leq \text{Sup}_{\zeta\epsilon W_\circ(G)} \text{Sup}_{\xi\epsilon W_\circ(H)}[|\zeta||\xi|\|E\|\|F\|] \tag{3.3.2}$$

Conversely, let $\{g_n\}_{n>1}$be a sequence of vectors of length one in a complex Hilbert space Z and let $\{h_n\}_{n>1}$ be a sequence of vectors of length one in a complex Hilbert space L.Define rank one operator,$(m \otimes g_n) \in B(Z)$ and $(f \otimes h_n) \in B(L)$ for a unit vector $\mathrm{m} \in Z$ and $\mathrm{g} \in L$ as $(m \otimes g_n)x = (x, g_n\rangle m$ and $(f, h_n)y = (y, h_n\rangle f$ $\forall$ $x \in Z$ and $\forall$ $y \in L$.

Define the $W_\circ(G)$ and $W_\circ(H)$ of G and H is defined as

$$W_\circ(G) = \{\zeta\epsilon\mathbb{C}: (Gg_n, g_n\rangle \to \zeta, \|g_n\| = 1 \ and\ \|Gg_n\| = \|G\|\}$$

and

$$W_\circ(H) = \{\xi\epsilon\mathbb{C}: (Hh_n, h_n\rangle \to \xi, \|h_n\| = 1 \ and\ \|Hh_n\| = \|H\|\}$$

Now $\forall$ $\zeta\epsilon Z$ and $\xi\epsilon L$, if $\zeta\epsilon W_\circ(G)$ $\forall$ $G \in B(Z)$ then a sequence $\{g_n\}_{n>1}$ of vectors of length one exists in Z such that;

(i) $\lim_{n\to\infty}(Gg_n, g_n\rangle = \zeta$
(ii) $\lim_{n\to\infty}\|Gg_n\| = \|G\|$

and if $\xi\epsilon W_\circ(H)$ $\forall$ $H \in B(L)$ then a sequence $\{h_n\}_{n>1}$ exists of vectors of length one in L such that

(i) $\lim_{n\to\infty}(H\square_{n,}\square_n\rangle = \xi$

(ii) $\lim_{n\to\infty}\|H\square_n\| = \|H\|$

By finite rank one operator, the basic elementary operator norm of is given as

$$\|O\{(m\otimes g_n)g_n\otimes(f\otimes\square_n)\square_n\| = \|O_{E\otimes F,G\otimes H}\,(m\otimes g_n)g_n\otimes(f\otimes\square_n)\square_n\|$$

$$= \|O_{E\otimes F,G\otimes H}\,(m\otimes g_n)\otimes(f\otimes\square_n)(\square_n\otimes g_n)\|$$

$$\le \|O_{E\otimes F,G\otimes H}\,(m\otimes g_n)\otimes(f\otimes\square_n)\|\|(\square_n\otimes g_n)\|$$

$$\le \|O_{E\otimes F,G\otimes H}\,(m\otimes g_n)\|\|f\|\|\square_n\|\|\square_n\|\|g_n\|$$

$$\le \|O_{E\otimes F,G\otimes H}\,\|\|m\|\|g_n\|\|f\|\|\square_n\|\|\square_n\|\|g_n\|$$

$\le \|O_{E\otimes F,G\otimes H}\|$ since the m, f are unit vectors and $g_n\ \square_n$ are unit sequences such that $\|f\| = 1$, $\|m\| = 1$, $\|\square_n\| = 1$ and $\|g_n\| = 1$.

Therefore;

$$\|O_{E\otimes F,G\otimes H}\,\| \ge \|O_{E\otimes F,G\otimes H}\,(m\otimes g_n)g_n\otimes(f\otimes\square_n)\square_n\|$$

$$\ge \|O_{E\otimes F,G\otimes H}\,\{(m\otimes g_n)g_n\otimes(f\otimes\square_n)\square_n\}\|$$

$$\ge \|E\otimes F\{(m\otimes g_n)g_n\otimes(f\otimes\square_n)\square_n\}, G\otimes H\|$$

$$\ge \|E\otimes F\{(m\otimes g_n)\otimes(f\otimes\square_n)\}g_n\otimes\square_n G\otimes H\|$$

$$\ge \|E\otimes F\{(m\otimes g_n)\otimes(f\otimes\square_n)\}G\otimes H(g_n\otimes\square_n)\|$$

$$\ge \|\{E(m\otimes g_n)Gg_n\}\otimes\{F(f\otimes\square_n)H\square_n\}\|$$

$$\ge \|\{(Gg_{n,}g_n\rangle Em\}\otimes\{(H\square_n,\square_n\rangle Ff\| \text{ thus}$$

$$\|O_{E\otimes F,G\otimes H}\,\| \ge \|\{(Gg_{n,}g_n\rangle Gm\}\otimes\{(H\square_n,\square_n\rangle Ff\| \tag{3.3.3}$$

by taking the limits both sides as $n\to\infty$ for the inequality $(3.3.3)$ and $\forall\ \zeta\epsilon Z$ and $\xi\epsilon L$, if $\zeta\epsilon W_\circ(G)\ \forall\ G\in B(Z)$ then $\exists$ a sequence $\{g_n\}_{n>1}$ of vectors of length one in Z such that

(i) $\lim_{n\to\infty}(Gg_{n,}g_n\rangle = \zeta$

(ii) $\lim_{n\to\infty}\|Gg_n\| = \|G\|$

and similarly, for every $\xi\epsilon W_\circ(H)\ \forall\ H\in B(K)$ then $\exists$ a sequence $\{\square_n\}_{n>1}$ of vectors of length one in L such that

(i) $\lim_{n\to\infty}(H\square_{n,}\square_n\rangle = \xi$

(ii) $\lim_{n\to\infty}\|Hg_n\| = \|H\|$

then we have;

$$\|O_{E\otimes F,G\otimes H}\| \geq \|\zeta Em \otimes \xi Ff\|$$

$$\geq \|\zeta Em\|\|\xi Ff\|$$

$$\geq |\zeta|\|E\|\|m\||\xi|\|F\|\|f\|$$

$\geq |\zeta||\xi|\|E\|\|F\|$ since the m, f are unit vectors such that $\|f\| = 1$ and $\|m\| = 1$.

This is true for any $\zeta \epsilon W_{\circ}(G)$, $\xi \epsilon W_{\circ}(F)$ and for any unit vector $m \epsilon Z, f \epsilon L$.Since ζ, ξ and the unit vectors are chosen arbitrarily, then we get the double supremum for the lower bound;

$$\|O_{E\otimes F,G\otimes H}\| \geq \text{Sup}_{\zeta \epsilon W_{\circ}(G)} \text{Sup}_{\xi \epsilon W_{\circ}(H)}[|\zeta||\xi|\|E\|\|F\|] \tag{3.3.4}$$

Therefore, from equation (3.3.2) and (3.3.4) we have,

$\|O_{E\otimes F,G\otimes H} \backslash B(Z \otimes L)\| = \text{Sup}_{\zeta \in W_{\circ}(G)} \text{Sup}_{\xi \in W_{\circ}(H)}\{|\zeta||\xi|\|E\|\|F\|\}$. []

Corollary 3.4

Let Z and L be a complex Hilbert space, $B(Z \otimes L)$ be the set of bounded linear operators on $Z \otimes l$.If for all $U \otimes V \in B(Z \otimes L)$ with $\|U \otimes V\| = 1$, then we have $\|O_{E\otimes F,G\otimes H}\| = \|O_{E,G}\|\|O_{F,H}\|$ where $O_{E,G}$ and $O_{F,H}$ are the basic elementary operators on $B(Z)$ and $B(L)$ respectively.

Proof

Since $\|O_{E,G}\| = |\zeta|\|G\|$ and $\|O_{F,H}\| = |\xi|\|H\|$.Now from theorem 3.3, we have

$\|O_{E\otimes F,G\otimes H}\| = \text{Sup}_{\zeta \epsilon W_{\circ}(G)} \text{Sup}_{\xi \epsilon W_{\circ}(H)}[|\zeta||\xi|\|E\|\|F\|] = \text{Sup}_{\zeta \epsilon W_{\circ}(G)}|\zeta|\|E\| \text{Sup}_{\xi \epsilon W_{\circ}(H}|\xi|\,\|F\|$ We can rearrange this as

$\|O_{E\otimes F,G\otimes H}\| = \text{Sup}_{\zeta \epsilon W_{\circ}(G)}|\zeta|\|E\| \text{Sup}_{\xi \epsilon W_{\circ}(H}|\xi|\,\|F\|$. Notice that $E, G \in B(Z)$ and $F, H \in B(L)$.Thus

$\|O_{E,G}\| = \underset{\zeta \epsilon W_{\circ}(G)}{\text{Sup}} |\zeta|\|E\|$ while $\|O_{F,H}\| = \text{Sup}_{\xi \epsilon W_{\circ}(H}|\xi|\|F\|$.

Then substituting, we obtain $\|O_{E\otimes F,G\otimes H}\| = \|O_{E,G}\|\|O_{F,H}\|$

4.0 Conclusion

In this paper, we have determined the lower bound of the norm of an elementary operator of length two in a tensor product using the Stampfli's maximal numerical range.

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