CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

EXAMINATION FOR THE AWARD OF DEGREE OF BACHELOR OF SCIENCE AND BACHELOR OF ARTS

MATH 448: STOCHASTIC PROCESSES

STREAMS: BSC & BA **TIME: 2 HOURS**

DAY/DATE: MONDAY 08/04/2024 **2.30 P.M – 4.30 P.M.**

INSTRUCTIONS:

- Answer question **ONE** and any other **TWO** questions.

QUESTION ONE (30 MARKS)

(a)State the four classifications of a stochastic process (4 marks)

(b)Determine the generating function A(s) for the sequence $\{ a_k, k = 0,1,2, . , . , .,\}$ where;

$(i) a_k = 2$ (3 marks)

$(ii) a_k = k^2$ (3 marks)

$(iii) a_k = \frac{1}{k!}$ (3 marks)

(b) A sequence of probabilities $\{p_k\}$ is such that $p_k = P(X = k)$

$$\text{If } p_k = P(X = k) = \begin{cases} \binom{n}{k} p^x q^k, \mathrm{k} = 0,1,2, . , . , . , \mathrm{n} \\ 0 \qquad\qquad\qquad , otherwise \end{cases}$$

where $q = 1 - p$,

(i) Determine the probability generating function of the sequence $\{p_k\}$ (3 marks)

(ii) Use the probability generating function of the sequence $\{p_k\}$ obtained above to find the mean and variance of X (5 marks)

(c)The probability generating function of a branching process is given by;

$$G(s) = \frac{1}{12} + \frac{2}{3}s + \frac{1}{4}s^2$$

Determine the probability of extinction of the process. (3 marks)

(d) A local bank has a single teller machine (ATM). Customers arrive at the ATM according to poisson input process at a mean rate of 20 per hour. The time a customer takes to conclude a transaction has an exponential distribution with a mean of 2.5 minutes. Determine;

(i) The average waiting time of a customer in the queue (2 marks)

(ii) The average number of customers in the system (2 marks)

(iii) The average queue length (2 marks)

QUESTION TWO (20 MARKS)

(a). Suppose X and Y are two independent random variables with respective probability generating functions $G_X(s)$ and$G_Y(s)$. Let Z be another random variable defined by;

$$Z = X + Y$$

If $G_Z(s)$is theprobability generating function of Z,

show using probability generating functions that,

(i) $G_Z(s) = G_X(s) + G_Y(s)$ (2 marks)

(ii) $E(Z) = E(X) + E(Y)$ (2 marks)

(iii) $Var(Z) = Var(X) + Var(Y)$ (4 marks)

(b) Let $Z_N = X_1 + X_2 + \ldots\ldots + X_N$, where $X_i(i = 1,2, \ldots, N)$ is a random variable and N is also a random variable. Assume that the $X_i^{'}\,s$ are independent and identically distributed as X and let $\phi(s)$, $H(s)$ and $G(s)$ denote the probability generating functions of Z_N, N and X respectively

Prove that,

(i) $\phi(s) = H[G(s)]$ (4 marks)

(ii) $E(Z_N) = E(X)E(N)$ (3 marks)

(iii) $Var(Z_N) = E(N)Var(X) + [E(X)]^2\, Var(N)$ (5 marks)

QUESTION THREE (20 MARKS)

(a) Let Z_n =The size of the n^{th} generation in the ordinary branching process.

And $G_n(s) = E(s^{Z_n})(n = 1,2,3,. \; . \; . \; . \; . \; .)$ be the probability generating function of Z_n.

Show that,

(i) $G_n(s) = G(s)[G_{n-1}]$ (4 marks)

(ii) $E(Z_n) = \mu^n$, where $\mu = E(Z_1)$ (3 marks)

(b) Suppose $\{X_t; t \geq 0\}$ is a birth-death process with birth rate $\lambda_n$and death rate μ_n.

(i) State the axioms of the process (4 marks)

(ii)Derive the basic difference differential equations of the process

(7marks)

(iii) Deduce from the results in (a), the difference differential equations of the poisson process

(2marks)

QUESTION FOUR (20 MARKS)

(a) Define the following;

(i) transient state of a Markov chain (2 marks)

(ii) Aclosed set of states of a Markov chain (2 marks)

(b) A matrix of transition probabilities is given by;

$$P = \begin{bmatrix} 0.3 & 0.5 & b-a \\ 0.4 & 0.3 & 0.3 \\ a+b & 0.1 & 0.3 \end{bmatrix}$$

Determine the values of a and b (3marks)

(b) (i) Amarkov chain has transition probability matrix P given by,

$$P = \begin{bmatrix} 0.34 & 0.26 & 0.40 \\ 0.44 & 0.32 & 0.14 \\ 0.61 & 0.22 & 0.17 \end{bmatrix}$$

Draw the transition diagram for the Markov chain (3 marks)

A Markov chain has the matrix of transition probabilities given by;

$$P = \begin{bmatrix} \frac{4}{9} & \frac{5}{9} \\ \frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

(i) Find the stationary distribution of the chain (4 marks)

(ii) State the value of,

$\lim_{n\to\infty} P^n$ (1 mark)

(c) A Markov chain has the matrix of transition probabilities given by;

$$P = \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

Assuming $X_0 = 1$, determine μ_{12},the mean passage time of the Markov chain (5 marks)

QUESTION FIVE (20 MARKS)

(a) Define the following as used in queuing models

(i) Arrival rate (λ)

(ii) Service rate (μ)

(iii) Traffic intensity (ρ) (3 marks)

(b)The steady state equations for the poisson queueing process are given by

$$P_{n+1} = \frac{(\lambda_n + \mu_n)P_n - \lambda_{n-1}\mu_{n-1}P_{n-1}}{\mu_{n+1}}$$

for $1 \leq n$

and

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 \text{ for } n = 0$$

Show that

$$P_0 = \left[1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \left(\frac{\lambda_{i-1}}{\mu_i}\right)\right]^{-1}$$

(7 marks)

(c) At a one-man barber shop, it takes on the average half a for a hair-cut, and customers arrive at an average rate of one per every 45 minutes. Assuming arrivals have a poisson distribution and service time has a negative-exponential, determine, on average, (10 marks)

(i) how many customers there will be in the shop

(ii) what time a customer will spend in the shop

(iii) of that time, what part will be spent waiting for service

(iv) what portion of the day the barber will be idle

(v) the probability that a customer has to wait for more than 30 minutes

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