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A NOTE ON QUASI-SIMILARITY OF OPERATORS IN HILBERT SPACES

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ABSTRACT
This paper reports on the notion of Quasi-similarity of bounded linear operators in Hilbert Spaces, defines a quasi-affinity from one Hilbert Space H to K and discusses results on quasi-affinities. It has been shown that on a finite dimensional Hilbert Space, quasi-similarity is an equivalence relation; it is reflexive, symmetric and transitive. Using the definition of commutants of two operators, an alternative result is given to show that quasi-similarity is an equivalence relation on an infinite dimensional Hilbert Space. The relationship between quasi-similarity and almost similarity equivalence relations in Hilbert Spaces using hermitian and normal operators is established.
Keywords: Quasi-similarity, Quasi affinities, Equivalence Relations, Commutants

INTRODUCTION
In this paper Hilbert spaces or subspaces will be denoted by capital letters, H and K respectively and T, A, B e.t.c. denotes bounded linear operators where an operator means a bounded linear transformation. $B(H)$ will denote the Banach algebra of bounded linear operators on H. $B(H, K)$denotes the set of bounded linear transformations from H to K, which is equipped with the (induced uniform) norm. If $T \in B(H)$, then T^*denotes the adjoint while $Ker(T)$, $Ran(T)$, $\overline{M}$ and $M^{\perp}$ stands for the kernel of T, range of T, closure of M and orthogonal complement of a closed subspace M of H respectively. For operator T, we also denote by $\sigma(T)$, $\| T \|$ the spectrum and norm of T respectively.

We need the following definitions:
An operator $T \in B(H)$ is said to be:

Self adjoint or Hermitian if $T^* = T$ (equivalently, if $\langle Tx, x\rangle \in \mathbb{R}$, $\forall \in H$);
Unitary if $T^*T = TT^* = I$; *Normal* if $T^*T = TT^*$(equivalently, if $\| Tx \| = \| T^*x \| \ \forall x \in H$).
Let H and K be Hilbert spaces. An operator $X \in B(H, K)$ is *invertible* if it is injective (one -to- one) and surjective (onto or has dense range); equivalently if $Ker(X) = \{0\}$ and $\overline{Ran(X)} = K$. We denote the class of invertible linear operators by $\mathcal{G}\ (H, K)$.
The *commutator* of two operators A and B, denoted by $[A, B]$ is defined by $AB - BA$.
The *self –commutator* of an operator A is $[A, A^*] = A^*A - AA^*$.
Two operators $T \in B(H)$ and $S \in B(K)$ are *similar* (denoted T $\approx$ S) if there exists an operator $X \in \mathcal{G}\ (H, K)$such that $XT = SX$ (i.e, $X^{-1}SX\ or\ S = XTX^{-1}$) where $\mathcal{G}\ (H, K)$ is a Banach subalgebra of B (H, K) which is an invertible operator from H to K.
Linear operators $T \in B(H)$ and $S \in B(K)$ are *unitarily equivalent* (denoted $T \cong S$), if there exists a unitary operator $U \in \mathcal{G}\ (H, K)$ such that $UT = SU$ $(i.e, T = U^*SU$ or equivalently $S = UTU^*)$.
Two operators are considered the *"same"* if they are unitarily equivalent, since they have the same properties of invertibility, normality, spectral picture (norm, spectrum and spectral radius).
An operator $X \in$ B (H, K) is *quasi-invertible* or *a quasi-affinity* if it is an injective operator with dense range (i.e. $Ker\ X = \{0\}$ and $\overline{Ran(X)} = K$; equivalently, $Ker\ X = \{\bar{0}\}$ and, $Ker\ X^* = \{\bar{0}\}$. Thus $X \in B(H, K)$ is quasi-invertible if and only if $X^* \in B(K, H)$ is quasi-invertible).
An operator $T \in B(H)$ is *a quasi –affine transform* of $S \in B(K)$ if there exists a quasi-invertible $X \in B(H, K)$ such that $XT = SX$ ($ie\ X\ intertwines\ T\ and\ S$). T is a *quasi-affine transform* of S if there exists a quasi-invertible operator intertwining T to S.
Two operators $T \in B(H)$ and $S \in B(K)$ are *quasi-similar* (denoted T$\sim$S) if they are quasi-affine transforms of each other (i.e., if there exists quasi-invertible operators $X \in B(H, K)$ and $Y \in B(K, H)$such that $TX = XS\ and\ YS = TY$).
T is said to be *densely intertwined* to S if there exists an operator with dense range intertwining T to S.
Two operators$S$and T are said to be *almost similar* (denoted by $S \overset{a.s}{\sim} T$) if there exists an invertible operator N such that the following two conditions are satisfied:

$$S^*S = N^{-1}(T^*T)N$$
$$S^* + S = N^{-1}(T^* + T)N.$$

Almost similarity of operators is also an equivalence relation.

MAIN RESULTS

2.1 Quasi − affinities of Operators

Definition 2.1.1: *The commutator* of $A \in B(H)$, $\{A\}'$ is the set of all operators in $\mathrm{B(H)}$ that commutes with A, i.e. $\{\mathrm{A}\}' = \{\mathrm{C} \in \mathrm{B(H)}: \mathrm{CA} = \mathrm{AC}\}$.
Proposition 2.1.2: *The commutant of an operator (is the set of all operators intertwining it to itself) intertwines itself.*
Claim: $\mathrm{C_1 + C_2} \in \{\mathrm{A}\}'$ and $\mathrm{C_1C_2} \in \{\mathrm{A}\}'$ whenever $\mathrm{C_1, C_2} \in \{\mathrm{A}\}'$.
Proof: $\{\mathrm{A}\}' = \{\mathrm{C} \in \mathrm{B(H)}: \mathrm{CA} = \mathrm{AC}\}$. Now $\mathrm{(C_1 + C_2)A = C_1A + C_2A = AC_1 + AC_2 = A(C_1 + C_2)}$, that is $\mathrm{(C_1 + C_2)A = A(C_1 + C_2)}$ and $\mathrm{(C_1C_2)A = C_1(C_2A) = C_1(AC_2) = (AC_2)C_1 = A(C_2C_1) = A(C_1C_2)}$ that is $\mathrm{(C_1C_2)A = A(C_1C_2)}$ as required.
Actually $\{\mathrm{A}\}'$ is an operator algebra which contains the identity.
Theorem 2.1.3.*Unitary equivalence is an equivalence relation.*
Proof: See [9].
Remark 2.1.4: *It has already been proved in [9] that similarity is an equivalence relation on* $B(H)$.

The natural concept of equivalence between Hilbert space operators is unitary equivalence which is stronger than similarity.

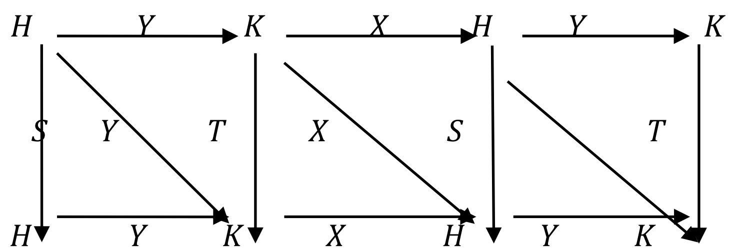
Theorem 2.1.5 [10, Proposition 3.3]: *If X is a quasi-affinity from H to K and Y is a quasi-affinity from K to L, then*

(a) *YX is a quasi-affinity from H to L and XY is a quasi-affinity from L to H.*

(b) *If $X \in B(H)$ is a quasi $-$ affinity, then X^* is a quasi-affinity.*

Proof: (a) Since S and T are called quasi-similar there exist quasi-affinities $X \in B(H,K)$ and $Y \in B(K,H)$ such that $XS = TX$ and $TY = YS$.

With this in mind, we draw the following diagram such that it "commutes".



We want to prove that XY and YX are quasi-affinities. Clearly, XY is one-to-one since it is the composition of one-to-one operators. It suffices to prove that XY has a dense range.

Note that $(XY) \subseteq H$. It follows that $\overline{XYH} = \overline{X(\overline{YH})} = \overline{X(K)} = H$. Therefore $\overline{Ran(XY)} = H$. This proves that XY has dense range.

Similarly, YX is one-to- one (since it is the composition of one-to-one operators). To show that it has dense range, note that $(YX) \subseteq K$. It follows that $\overline{YXK} = \overline{Y(\overline{XK})} = \overline{Y(H)} = K$. Therefore $\overline{Ran(YX)} = K$.

Now $S(XY) = XTY = (XY)S$, which shows that XY is a quasi-affinity in $\{S\}'$, the commutant of S.

Also, $(YX)T = Y(XT) = YSX = T(YX)$, that is, YX is a quasi-affinity in $\{T\}'$, the commutant of T.

(b) Since $X \in B(H)$ is a quasi-affinity, $Ker X = \{0\}$, $\overline{Ran(X)} = H$. We recall that

$$Ker\ X = Ran\ (X^*)^{\perp} \ldots\ldots\ldots (1)$$
$$Ker\ (X^*) = Ran\ (X)^{\perp} \ldots\ldots\ldots (2)$$
$$\overline{Ran(X)} = Ker\ (X^*)^{\perp} \ldots\ldots\ldots (3)$$
$$\overline{Ran(X^*)} = Ker\ (X)^{\perp} \ldots\ldots\ldots (4).$$

Therefore, since $Ker\ X = \{0\}$, we have $Ker\ (X)^{\perp} = H = \overline{Ran(X^*)}$ by (4) which implies that X^* has a dense range. $X^*$is one-to-one (since $Ker\ (X^*) = 0$). $X^*$is therefore a quasi-affinity.

Note: The proof of the following Theorem follows from Theorem 2.1.5.

Theorem 2.1.6 [10, Proposition 3.4]: *If A is a quasi-affine transform of B and B is a quasi-affine transform of C, then*

(a) A is a quasi-affine transform of C.

(b) $B^*$is a quasi-affine transform of A^*.

Proposition 2.1.7[10]: *If X is a quasi-affinity from H to K, then $|X| = \sqrt{X^*X}$ is a quasi-affinity on H (i.e. fromKto H). Moreover, $X|X|^{-1}$extends by continuity to a unitary transformation U from H to K.*

Lemma 2.1.8 [3, Lemma 2.6]: *Let $X \in B(H,K)$ and $Y \in B(K,L)$ be quasi-affinities where H, K and L are finite dimensional Hilbert spaces. Then the inverse $(YX)^{-1} \in B(L,H)$ of the composite YX exists and $(YX)^{-1} = X^{-1}Y^{-1}$.*

Proof: The operator $YK \in B(L,K)$ is bijective, so that YX exists. We thus have

$(YX)(YX)^{-1} = I_L$ is the identity operator on L. Applying Y^{-1} and using $Y^{-1}Y = I_K$, we obtain $Y^{-1}YX(YX)^{-1} = X(YX)^{-1} = Y^{-1}I_L = Y^{-1}$. Applying X^{-1} and using $X^{-1}X = I_H$ we obtain $X^{-1}X(YX)^{-1} = (YX)^{-1} = X^{-1}Y^{-1}$.

Proposition 2.1.9 [10, Proposition 3.4]: *If a unitary operator A on a Hilbert space H is the quasi-affine transform of a unitary operator B on a Hilbert space K then A and B are unitarily equivalent.*

Proof: Let $X \in B(H,K)$ be a quasi-affinity. Then

$XA = BX$ ……………………………………………………………………………...(1)

implies that $X = B^{-1}X = XA^{-1} = XA^*$ …………………………………………………….(2).

From (1) and (2) we obtain

$X|^2A = X^*XA = X^*BX = AX^*X = A|X|^2$ and by iteration $|X|^{2n}A = A|X|^{2n}(n = 0,1,\ldots)$; hence $p(|X|^2)A = A\,p(|X|^2)$ for every polynomial $p(x)$. Let $\{p_n(\mathrm{x})\}$ be a sequence of polynomials tending to $|X|^{\frac{1}{2}}$ uniformly on the interval $0 \le x \le \| X \|^{\frac{1}{2}}$. Then $p_n(|X|^2)$ converges (in the operator norm) to $|X|$so that we obtain a limit relation

$|X|A = A|X|$ …………………………………………..…………………(3).

From (1) and (3) it follows that $BU|X| = BX = XA = U|X|A = UA|X|$; because $|X|H$ is dense in H, it results that $BU = UA$. By Proposition 2.1.3 above U is unitary and hence A and B are unitarily equivalent.

Theorem 2.1.10: *Quasi-similarity is an equivalence relation on the class of all operators.*

Proof: Let $A \in B(H), B \in B(K), C \in B(L)$ respectively. First we show $A \sim A$.

Then $XA = AX$ and $AY = YA$ where X and Y are quasi-affinities. Choosing $X = Y = I$ (without loss of generality) we have that $A \sim A$. This proves reflexivity.

Now suppose that $A \sim B$. We show that $B \sim A$. Since $A \sim B$ there exist quasi-affinities $X \in B(H,K)$ and $Y \in B(K,H)$ such that $XA = BX$ and $BY = YA$. By symmetry of compositions, it is true that $BX = XA$and $YA = BY$. Hence $B \sim A$. This proves symmetry.

Suppose $A \sim B$ and $B \sim C$. Then we show that $A \sim C$.

There exist quasi- affinities $X \in B(H,K),\ Y \in B(K,H)$ and $Z \in B(K,L),\ R \in B(L,K)$ such that

$XA = BX,\quad AY = YB$ ……………………….……………………..…...(1)

and $ZB = CZ,\quad CR = RB$ ………………….………………………………………..(2).

$RZYX$ is a quasi-affinity; it is one-to-one since it is a composition of one-to-one operators.

$RZYXA = RZAYX$, since$YX \in \{A\}'$

$= RZYBX$, since $AY = YB$

$= RBZYX$, since $ZY \in \{B\}'$

$= CRZYX$, since $RB = CR$

Which is clearly a quasi-affinity and $AYXZR = YXAZR$, since $YX \in \{A\}'$

$= YBXZR$, since $XA = BX$

$= YXZBR$,since$XZ \in \{B\}'$

$= YXZRC$,since $ZR \in \{C\}'$.

Therefore $A \sim C$ .This proves that quasisimilarity is an equivalence relation.

Theorem 2.1.11: *If $T \in B(H)$ and $S \in B(K)$ are similar operators, then they are quasi-similar.*

Proof: There exist a quasi-invertible operator $X \in B(H,K)$ such that $XT = SX$.

This implies that $X^{-1}S = TX^{-1}$, where $X^{-1} \in B(K,H)$. $\Rightarrow S \sim T$.

2.2 RELATIONSHIP BETWEEN UNITARY EQUIVALANCE, QUASISIMILARITYAND ALMOST SIMILARIT

Proposition 2.2.1 [8, Proposition 1.2]: *If $A, B \in B(H)$ such that A and B are unitarily equivalent, then $A \overset{a.s}{\sim} B$.*

Proof: By assumption, there exists a unitary operator U such that $A = U^*BU$ which implies that $A^* = U^*B^*U$. Thus, $A^*A = U^*B^*UU^*BU = U^*B^*BU = U^{-1}B^*BU$,
and $A^* + A = U^*B^*U + U^*BU = U^*(B^* + B)U = U^{-1}(B^* + B)U$.

Proposition 2.2.2 [8, Proposition 1.3]: *If $A, B \in B(H)$ such that $A \overset{a.s}{\sim} B$, and if A is hermitian, then A and B are unitarily equivalent.*

Proof: By assumption, there exists an invertible operator N such that $A^* + A = N^{-1}(B^{*+}B)N$. Since A is hermitian and $A \overset{a.s}{\sim} B$ by Proposition 4.1.8 [7], B is hermitian. Thus we have $2A = N^{-1}2BN$ which implies that $A = N^{-1}BN$. This implies that A and B are similar ($i.e.\ A \sim B$) and since both operators are normal (both A and B are hermitian), they are unitarily equivalent.

Remark 2.2.3: *The Proposition 2.2.2 gives a condition under which almost similarity of operators implies similarity.*

Theorem 2.2.4: *If A is a normal operator and $B \in B(H)$ is unitarily equivalent to A, then B is normal.*

Proof: Suppose $B = U^*AU$ where U is unitary and A is normal. Then

$$B^*B = (U^*A^*U)(U^*AU) = U^*A^*AU = U^*AA^*U = B\,U^*A^*U = B\,U^*UB^* = BB^*$$

which proves the claim.

Corollary 2.2.5: *If $A, B \in B(H)$ are normal where H is an infinite dimensional Hilbert space such that A and B are Quasi-similar, then $A \overset{a.s}{\sim} B$.*

Proof: Since $A, B \in B(H)$ are quasi-similar, there exists quasi-affinities $X \in B(H, K)$ and $Y \in B(K, H)$ such that

$$XA = BX \text{ and } BY = YA \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

X and Y are both invertible and so XY, YX are both invertible. Without loss of generality, let $N = XY$ or YX. Then $XY \in \{A\}'$ and $YX \in \{B\}', i.e.\ AXY = XYA \Rightarrow A = XYA(XY)^{-1}$ and $YXB = BYX \Rightarrow B = (YX)^{-1}BYX$……………………………………………(2).

Since XY is invertible, $(XY)^* = Y^*X^*$ and $(XY)^{-1^*} = ((XY)^*)^{-1} = (Y^*X^*)^{-1} = X^{*^{-1}}Y^{*^{-1}}$.

Now, $A^*A = \left(X^{*^{-1}}Y^{*^{-1}}A^*Y^*X^*\right)XYA(XY)^{-1} = \left(X^{*^{-1}}Y^{*^{-1}}Y^*BX^*\right)XBYY^{-1}X^{-1}$
$= \left(X^{*^{-1}}BX^*\right)(XBX^{-1})$.

Since A and B are similar normal operators, they are unitarily equivalent by Proposition 2.2.2 so that $A^*A = \left(X^{*^{-1}}BX^*\right)XBX^{-1} = XB^*BX^{-1}$……………………………………………(3)

Also, $A^* + A = \left(X^{*^{-1}}BX^*\right) + (XBX^{-1}) = XB^*X^{-1} + XBX^{-1} = X(B^* + B)X^{-1} \ldots\ldots\ldots (4)$,
that is,
$A^*A = N^{-1}B^*BN$ and $A^* + A = N^{-1}B^* + BN$ where $N = X^{-1}$ is an invertible operator.

Remark 2.2.6: *Corollary 2.2.5 gives a condition under which similarity implies quasisimilarity which in turn implies almost similarity.*

The following Theorem enables us obtain an example of quasi-similar operators:

Theorem 2.2.7[8,Theorem 2.5]: *Suppose that for each α in some index set A, there are Hilbert spaces H_α and K_α and operators $T_\alpha \in B(H_\alpha)$ and $S_\alpha \in B(K_\alpha)$ respectively which are quasi-similar. Let T be the operator $T = \sum_{\alpha \in A} \oplus T_\alpha$ acting on the Hilbert space which is the direct sum of the spaces H_α and $S = \sum_{\alpha \in A} \oplus S_\alpha \in B(K)$ where $K = \sum_{\alpha \in A} \oplus K_\alpha$. Then T is quasi-similar to S.*

Proof: Suppose X_α and Y_α are the quasi-invertible operators such that $X_\alpha T_\alpha = S_\alpha X_\alpha$ and $T_\alpha Y_\alpha = Y_\alpha S_\alpha$. If $X = \sum_{\alpha \in A} \oplus X_\alpha / \| X \|$ and $Y = \sum_{\alpha \in A} \oplus Y_\alpha / \| Y \|$, then X and Y are the quasi-invertibles and satisfy the desired equations.

Example 2.2.8. *Let* A_n *and* B_n *be unilateral shift operators with weights* 1 *and* $\frac{1}{n}$ *respectively on an*n $-$*dimensional Hilbert space* H. *Then* A *is the Jordan canonical form for*B_n *and so* A *and* B_n *are similar. If* $A = \sum_{n=0}^{\infty} A_n$ *and* $B = \sum_{n=0}^{\infty} B_n$ *then by the above Theorem,* A *is quasi-similar to* B.
Remark 2.2.9: *Recall that an operator* $X \in B(H, K)$ *intertwines* $A \in B(H)$ *to*$B \in B(K)$ *if* $XA = BX$. *If* A *is densely intertwined to* B, *then there exists an operator with dense range intertwining* A *to* B.

Potential Conflicts of Interest
The authors declare no conflict of interest.

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