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Correspondence of Fixed-Point Theorem in $T_2, T_3 - SPACE$

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Abstract

Fixed-point theory (FPT) has lot of applications not only in the field of mathematics but also in various other disciplines. Fixed Point Theorem presents that if $T: X \to X$ is a contraction mapping on a complete metric space (X, d) then there exists a unique fixed point in X. FPT is also essential in game theory, in this case Brower Fixed Point has an application in game theory specifically in non-cooperative games and existence of Equilibrium. In particular, a game is a set of actions done by the participants defined by a set of rules. This is commonly described using mathematical concepts, which offers a concrete model to describe a variety of situations. On the other hand, the separation axioms T_i, $i = 0,1,2,3,4$ are vital properties that describes the topological spaces T_0, T_1, T_2, T_3 and T_4. It is noted that a $T_3 - space$ is a generalized version of T_2-space and since various results on application of fixed point theory in game theory on an arbitrary locally convex $T_2 -$ space has been established, in this study we sort to extend this concept to the general $T_3 - space$. The utilization of a symmetric property of Hausdorff space established that if two continuous commutative mappings are defined on a $T_3 - space$, then the two maps achieves unique fixed points.

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1 Introduction

A topological space X is said to be a Hausdorff space($T_2 -$ space) if $\forall\, a, b \in X, a \neq b$, then $\exists$ an open sets U and V such that $a \in U, b \in V, and\ U \cap V = \phi$, whereas a $T_3 - space$ is a regular $T_1 - space$ (Kuratowski, K. [1]. Furthermore, a topological space X is regular if $\forall a \in X$ and any closed set A of X , $\exists$ open sets $say\ U\ and\ V$ such that $a \in U, A \subseteq U\ and\ U \cap V = \emptyset$, while a topological space $X\ is$ said to be $T_1 - space$ if for any two points $a, b \in X$, where $a \neq b$ there exist open sets $U\ and\ V$ such that $a \in U, b \notin U$ and $b \in V,\ a \notin V$(Thron, W. J. [2]. In this study, we consider $T_i - spaces$ for $i = 0,1,2,3,4$ which are key separable topological spaces. In this case, $T_4 \to T_3 \to T_2 \to T_1 \to T_0$. A case of interest in our study is the result that a $T_3 -$ space implies $T_2 - space$ as illustrated by the following lemma.

Lemma 1.1: Every topological space $T_3 -$ space is a $T_2 -$ space Munkres et al., [3].

We also present of an important property of a symmetric Hausdorff space as presented by Gupta, V., Aydi, H., & Mani, N. [4] whereby, it is illustrated that a Hausdorff space X with a continuous mapping H is said to be a symmetric if it satisfies the following axioms;

I. $\mathrm{H(x, y) = 0}$ iff $\mathrm{x = y}$
II. $\mathrm{H(x, y) = H(y, x)}$
III. It is a $\mathrm{T_1}$ − space

Having seen that $T_3 - space$ is a generalization of $T_2 - space$ and suppose that the $T_3 - space$ satisfies a symmetric conditions, then as a consequence of Lemma 1.1 the following result is evident,

Lemma 1.2: Let X be a $T_3 - space\ and\ R{:}X \to X$ be a continuous mapping such $\forall x, y \in X$, then

I. $R(x, y) = 0 \quad iff\ x = y.$
II. $It\ is\ a\ regular\ T_1 - space$
III. $R(x, y) = R(y, x)$

Thus R is said to be symmetric on $T_3 - space$.

From lemma 1.1 and lemma 1.2 we have established $T_2 - space\ and\ T_3 - space$ are symmetric .

Popa, [5] established a generalized results of Banach Fixed point theorem through Hausdorff topological space by considering the properties below:

1. $H(x, y) \neq 0$ .... (1)
2. $H(Fx, Fy) \leq \alpha M(x, y) + \beta H(x, y)$ .... (2)
3. $H(Fx, Fy) \leq \alpha(\frac{H(x,Fx)H(y,Fy)}{H(x,y)} + \beta(H(x, y))$..(triangular inequality).......... (3)
4. $M(x, y) = max\,\{H(x, y), \frac{H(x,F(x)H(y,f(y)}{H(x,y)}\}$ .... (4)
5. $H^2(x, y) \geq H(x, x)H(y, y)$ .... (5)

Where $\alpha,\ \beta > 0\ and\ \alpha + \beta < 1$ for some $x_0 \in X$ such that $Fx_0 = x_1$. By defining the sequence x_n in a Hausdorff space X such that $Fx_n = x_{n+1}$ and $Fx_{n-1} = x_n$, then through iteration process the sequence $x_n = \{F^n x_0\}$ has a convergent subsequent. As a result fixed point of F is attain.

Based on the results discussed above by Popa, [5] and utilizing the concept of symmetric Hausdorff space, we similarly make an effort of derivation of Fixed Point Theorem under $T_3 - space$ in the proceeding proposition [6,7].

2 Main Results

Proposition 2.1:

Let $T: X \to X$ be a continuous mapping on $T_3 - space\ X$ into itself and let R: $X \to X$ be a continuous mapping which commutes with T satisfying the following conditions [8].

1. $R(x) \subset T(x)\ \forall\ x \in X$
2. $d(Rx, Ry) \leq \alpha(Tx, Ty), \forall\ x, y \in X,$

Then $T\ and\ R$ have a fixed point.

Proof:

Choosing $x_0 \in X$ such that $Rx_0 = Tx_1$. Based on this, we define a sequence $x_n\ in\ X$ such that
$Tx_n = R(x_{n-1})$

Step 1:

Let $x_0 \in X$ and x_1 be such that
$Tx_1 = Rx_0$, in general we choose x_n so that
$Tx_n = R(x_{n-1})$
$T(x_{n+1}, x_n) \leq \alpha(Tx_n, Tx_{n-1})$ For all n

It follows from property (2)

$T(x_n, x_{n+1}) \leq T(Rx_{n-1,}Rx_n) \leq \alpha T(x_{n-1}, x_n) + \beta T(x_{n-1}, x_n) < T(x_{n-1}, x_n) \ldots.$ (1)
Then from property (4)

$$T(x_{n-1}\,,\ x_n) = max\,\{T(x_{n-1}\,,\ x_n\,),\ \frac{T(x_{n-1}, Rx_{n-1})T(x_n, Rx_n)}{T(x_{n-1}, x_n)}\}$$
$$= max\{T(x_{n-1}, x_n), T(x_n, x_{n+1})\}$$

Suppose that

$T(x_n, x_{n+1}) > T(x_{n-1}, x_n)$, it follows from equation (1)
$T(x_n, x_{n+1}) \leq \alpha T(x_{n-1}, x_n) + \beta T(x_{n-1}, x_n) \ldots.$ (2)
Also if

$T(x_n, x_{n+1}) \leq T(x_{n-1}, x_n)$
It follows again from (I)
$T(x_n, x_{n+1}) \leq (\alpha + \beta)T(x_{n-1}, x_n) < T(x_{n-1}, x_n)$...(3)
From (2) and (3) we obtain
$\alpha T(x_n, x_{n+1}) + \beta T(x_{n-1}, x_n) \leq (\alpha + \beta)T(x_{n-1}, x_n)$
$\leq \alpha T(x_{n-1}, x_n) + \beta T(x_{n-1}, \mathrm{x_n})$
$\alpha T(x_n, x_{n+1}) + \beta T(x_{n-1}, x_n) \leq \alpha T(x_{n-1}, x_n) + \beta T(x_{n-1}, \mathrm{x_n})$
$\alpha T(x_n, x_{n+1}) \leq \alpha T(x_{n-1}, x_n)$
$T(x_n, x_{n+1}) \leq T(x_{n-1}, x_n) < T(x_{n-1}, x_n) \ldots..$ (4)

Using equation (3) and (4), and Repeating the process n times then,

$$T(x_n, x_{n+1}) < T(x_{n-1}, x_n) < \cdots < T(x_1, x_0) < T(x_0, x_1)$$

Since $\lim_{n\to\to\infty} x_n$ is bounded and letting $x_n\ \ \forall\ n \in N$ be a convergent sequence, denote the limit by t where $t \in X$. Let $x_{nk}\ \ \ \forall\ n \in N$ be subsequence such that if $\varepsilon > 0$ then by definition of convergence for $x_n\ as\ n \in N$,there exists $N \in \mathbb{N}$ such that $|x_n - t| < \varepsilon$ for $n \geq N$,but this value N will also work for x_{nk},this is because if $n \geq N$ then $x_{nk} = x_m$ for some $m \geq n \geq N$ and so $|x_{nk} - \mathrm{t}| = |x_m - \mathrm{t}\,| < \varepsilon$.Thus $|x_{nk} - \mathrm{t}| < \varepsilon$,as $n \to \infty, x_{nk} = t$

Hence, we obtain a monotone sequence which converge with all its subsequence to some real number $t \in X$.

Step 2

Next is to show t is a fixed point for T and R
Where $t \in X$ such that $Tx_n \to t, Rx_n \to t$
Since T is continuous it implies R is also continuous.
Since T and R commutes it follows that
$R(T(x_n) \to R(t) and\ \ T(R(x_n) \to T(t)$ So that
$R(T(x_n) = T(R(x_n))$
then
$T(t) = R(t)$

And

$T(T(t)) = T(R(t)) = R(R(t)$ (by commutativity)
From contraction mapping
$d(R(t), R(R(t)) \leq \alpha d(T(t), T(R(t)) = \alpha d(R(t), R(R(t))$
$d(R(t), R(R(t)) \leq \alpha d(T(t), T(R(t))$
$d(R(t), R(R(t)) - \alpha d(R(t), R(R(t)) \leq 0$,Since $\alpha \epsilon (0,1)$
$d(R(t), R(R(t))(1 - \alpha) \leq 0$
$R(t) = R(R(t)) = T(R(t))$, then $R(t)$ is a fixed point for T and R

Step3

To show that R and T have unique fixed point, we
Suppose $R(t) = T(t) = t$ and $R(t') = T(t') = t'$
Then it follows from contraction principle,
$d(t, t') = d(R(t), R(t')) \leq \alpha d(T(t), T(t')) = \alpha d(t, t')$
$but\ \alpha < 1$
And thus
$t = t'$
Thus R and T have unique fixed point ■

3 Conclusion

In this study, it has been established that the generalized result by Popa, [5] of Banach Fixed point theorem in a T_2 topological space can be extended to a T_3 space if the considered $T_3 - space$ possesses two continuous maps that commutates with one another.

Disclaimer (Artificial intelligence)

Authors hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

Competing Interests

Authors have declared that no competing interests exist.

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