CHARACTERIZATION AND COMMUTATION RELATIONS ON SQUARE NORMAL AND CLASS Q^* OPERATORS IN HILBERT SPACES

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DECLARATION AND RECOMMENDATION

Declaration

This thesis is my original work and has not been presented for a degree in any other university.

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Recommendation

This thesis has been examined, passed and submitted with our approval as university supervisors.

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DEDICATION

I dedicate this thesis to my family: Boniface, Tim Louis, and Charlotte. They have always been my pillars of strength and encouragement. Your unconditional love, sacrifices and unwavering belief in my abilities have constantly motivated me.

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ABSTRACT

Many researchers have widely studied operators in Hilbert spaces due to their wide application in areas like computer programming, financial mathematics and quantum physics. The study of operators in Hilbert space has been categorized according to their properties, the relation between different classes and their spectral properties. Researchers have studied various operators in Hilbert spaces, examining their algebraic properties, commutation relations, independence and inclusions. The classical normal operators has played an important role in the development, study and generalization of these classes of operators in Hilbert spaces. This study focused on the extension of properties of normal operators to two classes of operators, the square normal operators and class Q^* operators. The aim was to determine their characterization, algebraic properties and relationship with other operators in Hilbert space. By use of the relationships with normal operators, this study has established that for any square normal operator $T \in B(H)$, then T^*, T^{-1} and any other operator unitarily equivalent to T are square normal operators. Furthermore, it has been shown that for square normal operators T, S and scalar $\lambda \in \mathbb{C}$, then (λT), $(\lambda + T)$, $(T + S)$ and (TS) are square normal operators provided some certain conditions are met. The study shows that class Q^* operators are not convex and establishes that if two class Q^* operators T and S commute, then their sum $T + S$ is in class Q^* and their product TS is in class Q^* if $TS^* = S^*T$ and $T^*S = ST^*$. The research also found that 2-normal operators are both square normal and class Q^* operators while the class of 3-normal and square normal operators are independent. Furthermore, the study observed that while class Q^* operators are square normal, the converse is not necessarily true. These findings contribute to the existing knowledge in operators theory and functional analysis and offer potential applications in practical domains such as computer science, finance, and quantum mechanics.

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CHAPTER ONE
INTRODUCTION

1.1 Background Information

The concept of Hilbert Spaces, named after David Hilbert allows generalizing the methods of linear algebra and calculus from finite-dimensional Euclidean vector spaces to spaces that may be infinite-dimensional as seen in Berberian (1976). Hilbert spaces arise naturally and frequently in mathematics and physics as functional spaces. According to Rudin (1973), a Hilbert space (H) is a complete inner product space such that $\forall x, y, z \in H$ and $\beta, \alpha \in \mathbb{C}$, the following conditions are satisfied;

(i) $\langle \lambda x + \beta y, z\rangle = \lambda\langle x, z\rangle + \beta\langle y, z\rangle$

(ii) $\overline{\langle x, y\rangle} = \langle y, x\rangle$

(iii) $\langle x, x\rangle \geq 0$, $\langle x, x\rangle = 0$ if and only if $x = 0$

There are two types of Hilbert spaces; the finite-dimensional and the infinite-dimensional Hilbert spaces. This research will focus on the operators on finite-dimensional Hilbert spaces, which are the n-tuples of real numbers ($\mathbb{R}^n$), and the n-tuples of complex numbers ($\mathbb{C}^n$).
Weidmann (1980), noted that operators are special maps that make mathematical calculations and programming language possible. The most important classes of operators in Hilbert spaces are the self-adjoint, the unitary and the normal operators. The concept of the adjoint of an operator is extremely important in the study of linear operators in the Hilbert spaces. Furuta (2001) shows that every bounded operator has a unique adjoint and it extends the idea of complex conjugation. An illustrative example is provided by considering a scalar operator $T= \lambda I$. In this case the adjoint T^* of scalar operator λI is $\overline{\lambda} I$.
Indeed for every $x, y \in H$,

$$\begin{aligned}\langle \lambda x, y\rangle &= \lambda\langle x, y\rangle \\ &= \overline{\overline{\lambda\langle y, x\rangle}} \\ &= \overline{\overline{\lambda}\langle y, x\rangle} \\ &= \overline{\langle \overline{\lambda} y, x\rangle} \\ &= \langle x, \overline{\lambda} y\rangle\end{aligned}$$

The majority of classes of operators in $B(H)$ are defined according to the relation between operator T and its adjoint T^*.

The notion of normal operators (and more specifically normal matrices) is well studied by authors like Riyadh (2014). Normal operators generalize self-adjoint operators and unitary operators. The study of operators related to normal operators has been very successful in the sense that interesting results like the classical Putnam-Fuglede theorem has been obtained. Researchers have defined many classes of operators by either generalizing or relaxing the concept of normality. For instance, binormal operators are defined by generalizing the concept of normality. Rhaly (1994) defined posinormal operators by making them satisfy certain conditions of normal operators in the hope that some of the results which hold for normal operators will hold for these classes of operators. According to Rhaly (1994), the following are some of the properties of posinormal operators.

(i) Every self adjoint operator is posinormal. Consequently, every normal operator is posinormal.

(ii) Normal operators are posinormal but not all posinormal operators are normal.

Alzuraiqi and Patel (2010), introduced a class of n-normal operator where an operator $T \in B(H)$ is n-normal if $T^nT^* = T^*T^n$. In their study, Alzuraiqi and Patel (2010) established the relationship between the normal operators and the n-normal operators where any bounded normal operator is n-normal for any $n \in \mathbb{N}$. Alzuraiqi and Patel (2010) also noted that n-normal operators need not be normal. According to Alzuraiqi and Patel (2010), the following are characterizations of n-normal operators.

(i) If $T \in B(H)$ is an n-normal operator, then T^{-1}, if it exists and T^* are n-normal operators.

(ii) Any operator $S \in B(H)$ unitarily equivalent to an n-normal operator T, is n-normal.

(iii) If T, S are n-normal operators such that $TS = ST$, then the product TS is n-normal.

(iv) The sum $T + S$ of two commuting n-normal operators need not be n-normal operators.

(v) If T is an n-normal operator, then T^m is an n-normal operator for any positive integer m.

The class of 2-normal operators, denoted by $2N$, was enlarged by Jibril (2011) to the class of operators whose squares are 2-normal, SN. The author proved that the product of two commuting operators in SN is also in SN but their sum need not belong to SN. He showed that the direct sum and the tensor product of the operators in SN are also in SN. Jibril (2011), also studied the relationship of SN operators with parahyponormal and $3N$operators. He noted that these two classes of operators are independent with SN operators. The following are some of the characterizations of the SN operators as seen in Jibril (2011).

(i) Unitarily equivalence does not imply similarity. This is explained in the following example.

Consider operators T, S and X below

$$T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, S = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } X = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

acting on a two-dimensional Hilbert space $\mathbb{R}^2$. By direct calculation, $T \in SN$ but

$$S^* = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$S^{*2} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$S^2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$S^4 = S^2S^2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$S^4S^{*2} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$S^{*2}S^4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$S^4S^{*2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = S^{*2}S^4$$

and

$S = X^{-1}TX$ where $X^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

(ii) If $T \in SN$, then, $T + kI$ need not be in SN.

(iii) The class of SN is not convex.

To illustrate this, consider two operators $T, S \in B(H)$ where

T= $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Now,

$$T^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T^2T^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T^*T^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$T^2T^* = T^*T^2$ hence 2-normal.

Also,

$$S^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S^2 S^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S^* S^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$S^2 S^* = S^* S^2$ hence 2-normal.

The operators T and S are 2-normal thus in SN.

Now by letting operator

$$A = \frac{1}{2}T + \frac{1}{2}S = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix},$$

then by direct calculation, one can show that

$$A^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$A^4 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{16} & \frac{1}{4} \\ 0 & \frac{1}{16} \end{bmatrix}$$

$$A^* = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$A^{*2} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$A^4 A^{*2} = \begin{bmatrix} \frac{1}{16} & \frac{1}{4} \\ 0 & \frac{1}{16} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{9}{64} & \frac{1}{16} \\ \frac{1}{32} & \frac{1}{64} \end{bmatrix}$$

$$A^{*2}A^4 = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{16} & \frac{1}{4} \\ 0 & \frac{1}{16} \end{bmatrix} = \begin{bmatrix} \frac{1}{64} & \frac{1}{16} \\ \frac{1}{32} & \frac{9}{64} \end{bmatrix}$$

Clearly, $A^4A^{*2} \neq A^{*2}A^4$. Thus $A \notin SN$ which means SN is not convex.

Jibril (2010), introduced a class Q of operators acting on Hilbert space H, that is, for any $T \in Q$, $T^{*2}T^2$ =$(T^*T)^2$ and investigated the basic properties of this class of operators. Jibril (2010) showed that quasinormal and isometric operators are in class Q, gave conditions under which class Q operators become quasinormal and also showed that not all class Q operators are isometric. Jibril (2010) gave several characterizations of 2-normal operators such that T is 2-normal if and only if $T^2T^* = T^*T^2$. Jibril (2010) also showed that class $2N$ of 2-Normal operators and class Q are independent.

Panayappan *et al.* (2012) introduced n power class Q operators where an operator $T \in B(H)$ is n- power class Q if $T^{*2}T^{2n} = (T^*T^n)^2$ and gave some basic properties of these operators. Panayappan *et al.*(2012), noted that n power class Q operators need not be normal. This can be demonstrated in the following example.

Let $T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ be an operator acting on a three-dimensional Complex Hilbert space.

Then T is 2- power class Q but it is not 2-normal and hence not normal. According to the authors, any n normal operator belongs to n power class Q. Panayappan *et al.*(2012) also found that the sum and the product of 2 power class Q are not 2 power class Q as in the following example.

Consider the operators $S = \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}$ and $T = \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix}$.

These operators are 2- power class Q but $S+T$ and ST are not 2- power class Q. Panayappan *et al.* (2012) also noted that 2- power class Q and 3- power class Q are independent.

Recently, the class Q^* operator where for $T \in B(H)$, $T^{*2}T^2 = (TT^*)^2$ was introduced by Wanjala and Nyongesa (2021). It is noted that any operator $T \in B(H)$ which is in this class also belongs to a class of square normal operators that were introduced by Mahmood (2016). Since square normal operators are extensions of normal operators, and every normal operator is n-normal, the commutation relations of class Q^* operators with normal and n-normal operators were established in this study. This study also focused on the characterization of square normal operators and class Q^* operators and established a relationship between these two classes of operators.

1.2 Statement of the Problem

Recent studies in operator theory have identified two classes of operators in Hilbert spaces - the square normal and class Q^* operators. However, these two classes of operators have yet to receive substantial attention in terms of their characterization and commutation relations. The scarcity of literature in these areas hinders the ability to utilize their applications in mathematical and other applied disciplines. Moreover, the commutation relations between these operators and other existing classes of operators remain largely unknown. This study investigated the properties of these two classes of operators and explored the commutation relations between them and n-normal operators in Hilbert spaces.

1.3 Objectives of the Study

1.3.1 Broad Objective

The broad objective of this study was to determine the characterization and commutation relations of square normal and class Q^* operators as an extension of classical normal operators in Hilbert spaces.

1.3.2 Specific Objectives

The specific objectives of this study are to:

(i) To determine the properties of square normal operators with a focus on their scalar multiplication and scalar addition, sum and product of two commuting and non-commuting square normal operators.

(ii) To determine some properties of class Q^* operators such as cartesian decomposition, sum and product of two commuting class Q^* operators and the direct sum and tensor product of operators in class Q^*.

(iii) To establish the relation between square normal and class Q^* operators and also with operators such as normal and n-normal operators.

1.4 Significance of the Study

The findings in this study will make a significant contribution to both theoretical and applied aspects of operators in Hilbert spaces. In a theoretical approach, the study will contribute to the foundations of the operator theory and enrich our understanding of the characterization of the square normal and class Q^* operators in Hilbert spaces. The systematic investigation of commutation relations will be valuable for researchers working with operators and will provide them with more understanding of the relationships and interactions between operators. By addressing the gaps in the current literature and focusing on areas that have not been extensively studied, this study has established a bridge between the existing knowledge and the unexplored territories encouraging further exploration and dialogue among researchers. The study also has enhanced the potential application of Square normal and class Q^* operators in fields such as computer programming, financial mathematics and quantum physics. In computer programming, the study will help in creating complex codes for security encryption. It will also help in making models for generating better security numbers that are not biased, nonrepetitive, and unpredictable. In the field of financial mathematics, these results will help in creating mathematical models which in return will assist managers in making objective decisions for maximum profitability and sustainability of the business. The findings on these operators will also help in creating an arbitrage pricing model that can identify inputs that might influence an asset in business. These results will also be used in probability theory by helping in modeling complex conditions such as predicting drug-drug interactions in pharmacology.

1.5 Definitions and Terminologies

In this section, we will provide definitions and relevant terminologies used in the study.

Definition 1.5.1: Inner product. (Furuta, 2001)

Let X be a non-empty set. An inner product is the function
$\langle , \rangle : X \times X \longrightarrow \mathbb{C}$ which satisfies the following axioms:

(i) $\langle x, x\rangle \geq 0$ with $\langle x, x\rangle = 0$ iff $x = 0$.

(ii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ $\forall x, y \in X$.

(iii) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ $\forall x, y, z \in X$.

(iv) $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ $\forall x, y \in X$ and all $\lambda \in \mathbb{C}$.

Definition 1.5.2: Hilbert space. (Furuta, 2001)

Hilbert space is a complete inner product space.

Definition 1.5.3: Linear operator. (Zhu, 1993)

A mapping $T : H \longrightarrow H$ is said to be a linear operator if it satisfies the following:

(i) *Additive*: $T(x + y) = T(x) + T(y), \forall x, y \in H$.

(ii) *Homogeneous*: $T(\lambda x) = \lambda T(x), \forall x, y \in H$ and all $\lambda \in \mathbb{C}$.

Definition 1.5.4: Bounded linear operator. (Halmos, 1978)

A linear operator T on a Hilbert space H is said to be bounded if there exist a constant $c > 0$ such that $\|Tx\| \leq c\,\|x\|\,\forall x \in H$.

Definition 1.5.5: Normed space. (Berberian, 1974)

A normed space is a vector space X (assumed to be over the complex field $\mathbb{C}$) equipped with a norm $\|.\|$ satisfying:

(i) $\|x\| \geq 0$.

(ii) $\|x\| = 0$ if and only if $x = 0$.

(iii) $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{C}$

(iv) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Remark 1.5.6: (Bognar, 2012)

The inner product can also be defined in terms of the norm, that is, $\langle x, x\rangle = \|x\|^2$.

Definition 1.5.7: Adjoint of operator. (Kreyszig, 1989)

Let H be a Hilbert space $T : H \to H$ be a bounded linear operator. The bounded linear operator $T^* : H \to H$ is defined by $\langle y, Tx\rangle = \langle T^*y, x\rangle \; \forall x, y \in H$ is called the adjoint operator T.

Definition 1.5.8: Self adjoint operator. (Gheondea, 2009)

Let H be a Hilbert space and $T : H \to H$ be a bounded linear operator. T is self-adjoint if $T^* = T$.

Definition 1.5.9: Invertible operator. (Weidmann, 1980)

Let H be a Hilbert space and $T : H \to H$ be a bounded linear operator. T is invertible with inverse S if there exist $S \in B(H)$ such that $ST = I = TS$ where $I \in B(H)$ is the identity operator.

Definition 1.5.10: Normal operator. (Alzuraiqi and Patel, 2010)

Let $T \in B(H)$. T is said to be normal if it commutes with its adjoint,
i.e $(T^*T = TT^*)$, equivalently $T^*T - TT^* = 0$.

Definition 1.5.11: Positive operator. (Alzuraiqi and Patel, 2010)

An operator $T \in B(H)$ is said to be positive if $T^* = T$ and $\langle Tx, x\rangle \geq 0 \; \forall x \in H$.

Definition 1.5.12: n-power normal operator. (Jibril, 2008)

Let $T \in B(H)$. Then, T is n-power normal if $T^nT^* = T^*T^n \; n \in \mathbb{N}$.

Definition 1.5.13: Hyponormal operator. (Panayappan *et al.,* 2012)

Let $T \in B(H)$. Then, T is hyponormal if $TT^* \leq T^*T$.

Definition 1.5.14: Class Q operator. (Jibril, 2010)

Let $T \in B(H)$. For any $T \in Q$, $T^{*2}T^2 = (T^*T)^2$.

Definition 1.5.15: Class Q^* operator. (Wanjala and Nyongesa, 2021)

An operator $T \in B(H)$ is called a class Q^* if $T^{*2}T^2 = (TT^*)^2$.

Definition 1.5.16: Quasi-class Q. (Devika and Suresh, 2013)

An operator $T \in B(H)$ is Quasi-class Q if $T^{*3}T^3 - 2T^{*2}T^2 + T^*T \geq 0$.

Definition 1.5.17: n-power-hyponormal operators. (Messaoud and Mostefa, 2016)

An operator $T \in B(H)$ is called an n-power-hyponormal operator if $T^nT^* \leq T^*T^n$.

Remark 1.5.18:

This class includes all normal, all n-normal and all hyponormal operators.

Definition 1.5.19: Square normal operator. (Mahmood, 2016)

An operator $T \in B(H)$ is square normal if $T^2(T^*)^2 = (T^*)^2T^2$.

Definition 1.5.20: Numerical range. (Furuta, 2001)

For an operator $T \in B(H)$. The numerical range of T is the set

$W(T)$=$\{\lambda \in \mathbb{C} : \lambda = \langle Tx, x\rangle, \|x\| = 1, x \in H\}$.

Definition 1.5.21: Spectrum. (Kreyszig, 2001)

Let H be a complex Hilbert space with inner product and T a bounded linear operator on H.

The set, $\delta(T)$ = $\{\lambda \in \mathbb{C}$: $T - \lambda I$ is not invertible$\}$ is referred to as the spectrum of T.

Definition 1.5.22: Binormal operator. (Panayappan and Sivamani, 2012)

An operator $T \in B(H)$ is Binormal if T^*T commutes with TT^*.

That is, $(T^*T)(TT^*) = (TT^*)(T^*T)$.

Definition 1.5.23: class μ operator. (Jibril, 2013)

An operator $T \in B(H)$ is in class μ if $T^2 = -T^{*2}$.

Definition 1.5.24: n power class Q operator. (Panayyapan and Sivanami)

An operator $T \in B(H)$ is n power class Q if $T^{*2}T^{2n} = (T^*T^n)^2$.

CHAPTER TWO

LITERATURE REVIEW

2.1 Characterization and Properties of Square Normal Operators

Several researchers have dealt with the characterization and properties of normal operators in a Hilbert space. Putnam (1951) showed that normal operators are closed under scalar multiplication and obtained theorem 2.1.1

Theorem 2.1.1: Putnam, (1951)

Let $T \in B(H)$ be a normal operator on a Hilbert space H. Then the T is closed under scalar multiplication, that is, if T is normal, then λT is a normal operator $\forall \lambda \in \mathbb{C}$.

To illustrate this, consider a normal operator T and a scalar $\lambda \in \mathbb{C}$

Now,

$$\begin{aligned}(\lambda T)(\lambda T)^* &= (\lambda T)(\overline{\lambda} T^*)\\ &= \lambda\overline{\lambda}(TT^*)\\ &= (\overline{\lambda}\lambda(T^*T)\\ &= (\overline{\lambda}T^*)(\lambda T)\\ &= (\lambda T)^*(\lambda T)\end{aligned}$$

Therefore T is closed under scalar multiplication.

Since square normal operators are an extension of normal operators, this study extended this concept of scalar multiplication and scalar addition to square normal operators.

The commutation relation $TT^* = T^*T$ is a fundamental property of normal operators since it plays a crucial role in the spectral theorem for normal operators. According to the spectral theorem, if T is a normal operator, then it can be diagonalized by a unitary operator. Embry (1970) showed a normal operator T must commute with its adjoint and for the spectral properties, 0 must not be in the numerical range of the operator T. This condition ensures that the operator T is not singular. The two conditions for normality of operators are stated in Corollary 2.1.2 below.

Corollary 2.1.2 : Embry, (1970) Corollary 1

If TT^* and T^*T commute and $0 \notin W(T)$, then T is normal.

Embry (1970), extended these conditions of normality of operator T to operator T^2 as seen in proposition 2.1.3

Proposition 2.1.3: Embry, (1970) Corollary 3

If T^2 is normal, and $0 \notin W(T)$, then T is normal.

Thus T^*T^2 and T^2T^* must commute and Corollary 2.1.2. is applicable. It is noted that the condition $0 \notin \delta(T)$ is not sufficiently strong to guarantee that T is normal when T^2 is normal. For example, for any non-normal square root of the identity operator I, $T^2 = I$ but T is not normal. However, if T^2 is normal and $\delta(T) \cap \delta(-T)= \emptyset$, then T is normal. This condition provides a more robust criterion for ensuring the normality of T compared to individual properties like $0 \notin \delta(T)$.

This notion of commutation of normal operators was extended to the sum and the product of two normal operators by Conway (1985). Theorem 2.1.4 gives the results of two normal operators each commuting with its adjoint.

Theorem 2.1.4: Conway, (1985)

If T and S are normal operators on H with the property that each commutes with the adjoint of the other, then

(i) $T + S$ is normal.

Since T and S are normal,then $TT^* = T^*T$ and $SS^* = S^*S$.

Also given that each operator commutes with the adjoint of the other, then $TS^* = S^*T$ and $ST^* = T^*S$.

Now we show that $(T + S)$ $(T + S)^*$ = $(T + S)^*(T + S)$

$$
\begin{aligned}
(T+S)(T+S)^* &= (T+S)(T^*+S^*) \\
&= TT^* + ST^* + TS^* + SS^* \\
&= T^*T + T^*S + S^*T + S^*S(\text{ Since } TS^* = S^*T \text{ and } ST^* = T^*S) \\
&= T^*(T+S) + S^*(T+S) \\
&= (T^*+S^*) + (T+S) \\
&= (T+S)^*(T+S)
\end{aligned}
$$

as required.

(ii) TS is normal.

To show $(TS)(TS)^* = (TS)^*(TS)$,

$$
\begin{aligned}
(TS)(TS)^* &= (TS)(S^*T^*) \\
&= T(SS^*)T^* \\
&= T(S^*S)T^* \\
&= (TS^*)(ST^*) \\
&= (S^*T)(T^*S) \\
&= S^*(TT^*)S \\
&= S^*(T^*T)S \\
&= (S^*T^*)(TS) \\
&= (TS)^*(TS)
\end{aligned}
$$

as required.

However, if the two normal operators T and S do not commute, then $T+S$ and TS are not necessarily normal.

This study aimed to determine if the sum and product of two commuting square normal operators are also square normal operators.

Mahmood (2016) extended normal operators to a broader category of square normal operators and stated Proposition 2.1.5 below.

Proposition 2.1.5 : Mahmood, (2016)

Let $T \in B(H)$. If T is a normal operator, then it is a square normal operator.

This can be demonstrated as follows

$$\begin{aligned} T^2(T^*)^2 &= TTT^*T^* \\ &= TT^*TT^* \\ &= T^*TT^*T \\ &= T^*T^*TT \\ &= (T^*)^2T^2 \end{aligned}$$

The converse of this statement is not true. The example below shows the existence of a square normal operator which is not normal.

Let $T = \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix}$, Then $T^* = \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix}$

$$T^2 = \begin{bmatrix} -i & 0 \\ i & -i \end{bmatrix} \begin{bmatrix} -i & 0 \\ i & -i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$(T^*)^2 = \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T^2(T^*)^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$(T^*)^2T^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $T^2(T^*)^2 = (T^*)^2T^2$, T is a square normal operator.

$$TT^* = \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix} \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$T^*T = \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$TT^* \neq T^*T$ and therefore T is not normal.

Proposition 2.1.6: Mahmood, (2016)

Let $T \in B(H)$. Then T is a square normal operator if and only if T^2 is normal.

This can be illustrated well by the definition of square normal operators and working on the adjoint of the operator.

Let T be a square normal operator, so

$$\begin{aligned} T^2(T^*)^2 &= (T^*)^2T^2 \\ &= T^2(T^2)^* \\ &= (T^2)^*T^2 \end{aligned}$$

This implies that T^2 is normal.

Fuglede-Putnam Theorem is a powerful tool that illuminates certain intriguing properties associated with the interplay between operators. Originally formulated for normal operators, this theorem has since found applications and extensions in various operator classes. Fuglede-Putnam Theorem is as stated in Theorem 2.1.8 below.

Theorem 2.1.7: Riyadh, (2014)

Let $T, N, M \in B(H)$ such that M and N are normal, then

$TN = MT$ implies $TN^* = M^*T$.

Given operators N, M and T as

$$N = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then,

$$TN = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$MT = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$N^* = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$M^* = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$TN^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$$M^*T = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

Note that square normal operators do not imply the normality of the operators, and as a result, the Fuglede-Putnam theorem cannot be generalized.

2.2 Characterization and Properties of Class Q^* operators

In Operator theory, there are various classes of operators, each with its unique set of properties and behaviors. Many researchers have studied class Q operators to understand their properties and their relation with other classes of operators. Wanjala and Nyongesa (2021) studied class Q^* operators and showed that this class is different from class Q.

As seen in Proposition 2.2.1, when you square an operator T and then take its adjoint T^*, or when you first take the adjoint and then square, a surprising consistency emerges and this symmetry is expressed as in Proposition 2.2.1

Proposition 2.2.1: Brown, (1953)
Let T be an operator. Then $\|T^*T\| = \|TT^*\| = \|T^*\|^2$
Hamos(1978) gave a consequence of proposition 2.2.1 as seen in the Corollary 2.2.2 below.

Corollary 2.2.2: Halmos, (1978)
Let T be an operator. Then

(i) $\|T^*T\| = \|TT^*\| = \|T\|^2$

(ii) $T^*T = 0$ if and only if $T = 0$

Proposition 2.2.1 and Corollary 2.2.2 give a basis for Corollary 2.2.3.

Corollary 2.2.3: Kreyszig, (1989)
Let T be an operator on a Hilbert space H. Then T^* is also an operator on H and the following properties hold:

(i) $\|T^*\| = \|T\|$

(ii) $(T^*)^* = T$

From (i), for $x, y \in H$, we know that $\langle x, T^*y\rangle = \langle Tx, y\rangle$
Let $T^*y = x$, then $\langle T^*y, T^*y\rangle = \langle TT^*y, y\rangle$
$\|T^*y\|^2 = \|TT^*y, y\|\|y\| \leq \|T\|\|T^*y\|\|y\|$
$\|T^*y\| \leq \|T\|\|y\|$. This implies that $\|T^*\| = \|T\|$

From (ii), given that $\langle x, Ty \rangle = \langle T^*x, y \rangle = \langle x, (T^*)^*y \rangle = \langle x, T^{**}y \rangle$
This implies that $Ty = T^{**}y$ implying that $T = T^{**}$

With $\|T^*\| = \|T\|$ of corollary 2.2.3 (i), $\|T^*T\| \leq \|T^*\|\|T^*\| = \|T^*\|^2$ hence $\|T^*T\| \leq \|T^*\|^2$.

According to Jibril (2010), operators belonging to class Q stay closed upon the addition of real constants. It's worth noting that if we have an operator T in the set Q, then any operator obtained by adding a real number to T will also belong to Q. This mathematical statement is presented in Proposition 2.2.4 below.

Proposition 2.2.4: Jibril, (2010)
If $T \in Q$, then so is $T + \lambda I$ for every real λ.

This is proved by contradiction.
Suppose that $T + \lambda I \notin Q$, then

$$((T + \lambda I)^*(T + \lambda I))^2 - (T + \lambda I)^{*2}(T + \lambda I)^2 \neq 0$$

Simplifying this,

$$T^{*2}T + T^*T^2 + \lambda IT^*T - T^*TT^* - TT^*T - \lambda ITT^* \neq 0 \tag{2.1}$$

Multiplying (2.1) on the right by T becomes,

$$T^{*2}T^2 + T^*T^3 + \lambda IT^*T^2 - (T^*T)^2 - TT^*T^2 - \lambda ITT^*T \neq 0 \tag{2.2}$$

Using the fact that $T^{*2}T^2 = (T^*T)^2$, equation (2.2) becomes

$$T^*T^3 + \lambda IT^*T^2 - TT^*T^2 - \lambda ITT^*T \neq 0 \tag{2.3}$$

Multiplying (2.3) on the left by T^* gives,

$$T^{*2}T^3 + \lambda IT^{*2}T^2 - T^*TT^*T^2 - \lambda I(T^*T)^2 \neq 0$$

which implies that

$$T^{*2}T^3 - T^*TT^*T^2 \neq 0 \tag{2.4}$$

Multiplying (2.4) on the left by T^*,

$$T^{*3}T^3 - T^{*2}TT^*T^2 \neq 0,$$

which implies that

$$T^*T^{*2}T^2T - T^*(T^*T)^2T \neq 0.$$

A contradiction. Thus $T + \lambda I \in Q$.
In this study, this concept of scalar addition and scalar multiplication was extended to class Q^* operators.

Proposition 2.2.5: Jibril, (2010)
If $T \in Q$, then $T \upharpoonright M$ of T to any closed subspace M of H that reduces T is also in Q.

We have $(T \upharpoonright M)^{*2}(T \upharpoonright M)^2 = (T^{*2} \upharpoonright M)(T^2 \upharpoonright M)$
$= (T^{*2}T^2) \upharpoonright M = (T^*T)^2 \upharpoonright M = (T^*T \upharpoonright M)^2 = (T \upharpoonright M)^{*2}(T \upharpoonright M)^2$.
Thus $T \upharpoonright M \in Q$.

In their recent research, Wanjala and Nyongesa (2021) have presented a compelling set of conditions within the framework of bounded linear operators on a Hilbert space. Proposition 2.2.6 describes a set of conditions that preserve certain properties under operations such as squaring, taking inverses, and unitary equivalence. These findings contribute to a better understanding of the behavior of operators in class Q^*.

Proposition 2.2.6: Wanjala and Nyongesa, (2021)
Let $T \in B(H)$, if $T \in Q^*$, then the following hold

(i) T^{*2} is in Q^*.

(ii) T^{-1} is in Q^* provided it exist.

(iii) Any $S \in B(H)$ that is unitarily equivalent to T is also in Q^*.

From (i), Since $T \in Q^*$, $T^2 \in Q^*$ hence $(T^2)^* = T^{*2} \in Q^*$.
From (ii), Since $T \in Q^*$ implies that $T^2 \in Q^*$. Thus $(T^2)^{-1} = (T^{-1})^2 \in Q^*$.
From (iii), Suppose $T \in Q^*$ and S is unitarily equivalent to T. Then there exists a unitary operator $U \in B(H)$ such that $S = U^*TU$ which implies that
$S^* = (U^*TU)^* = U^*T^*U$.
Thus

$$\begin{aligned} S^{*2}S^2 &= (U^*T^*U)^2(U^*TU)^2 \\ &= U^*T^*UU^*T^*UU^*TUU^*TU \\ &= U^*T^*UU^*T^*UU^*TUU^*TU \\ &= U^*T^{*2}T^2U \end{aligned}$$

$$\begin{aligned} (SS^*)^2 &= (U^*TUU^*T^*U)^2 \\ &= (U^*TT^*U)^2 \\ &= U^*TT^*UU^*TT^*U \\ &= U^*TT^*TT^*U \\ &= U^*(TT^*)^2U. \end{aligned}$$

Since $T^{*2} \in Q^*$, then $S^{*2} \in Q^*$ hence $S \in Q^*$.

This study determined whether similarity can replace unitary equivalence in proposition 2.2.6 (iii) mentioned above. Additionally, the study investigated whether the sum and product of two commuting operators in class Q^* are also in class Q^*.

2.3 Relation between Square Normal, Class Q^* and other Operators

In Hilbert spaces, various classes of operators often reveal profound connections and insights when they interact with each other. One such interesting set of relationships involves the trio of square normal operators, operators in the class Q*, and their interactions with other classes of operators in Hilbert space. By exploring this intricate web of relationships, we can not only deepen our understanding of each class but also uncover the harmonies and symmetries that exist between them.

Proposition 2.3.1: Jibril, (2010) Proposition 19

If $T \in B(H)$ is isometry, then $T \in Q$.

Since T is isometric, $T^*T = I$ which implies that
$(T^*T)^2 = I$ and $T^{*2}T^2 = T^*T = I$. Thus $T \in Q$.

In this study, investigations on the relationship between n-normal operators, class Q* and square normal operators was done.
Several researchers have explored the commutation relationships between two operators that belong to the same class and any bounded operator within a Hilbert space. These interactions are of significant importance as they have led to the discovery of several theorems, including the Fuglede-Putnam theorem, which is outlined in theorem 2.1.7. The following is an example of such an interaction between two square normal operators and a bounded operator.

Theorem 2.3.2: Mahmood, (2016)
Let S and T be two square normal operators. For any bounded linear operator A if

$$AS^2 = T^2A$$

Then

$$A(S^*)^2 = (T^*)^2A$$

Here, the algebraic properties of two commuting square normal operators were studied.
According to Jibril (2011) , if two operators T and S commutes, then their product $TS \in SN$.
However, if the product is replaced by the sum, then the result is not in general true. Also if $TS \neq ST$, then the result is not necessarily true. This study extended this property to the adjoint of operators to class Q^* operators.

Theorem 2.3.3: Mahmood, (2016)
Let S and T be two square normal operators which commute and let A be any bounded operator for which $0 \notin W(A)$. If $AS^2 = T^2A$, then S^2=T^2.

According to Wanjala and Nyongesa (2021), If $T \in Q^*$ such that T^2 is normal then T is normal. This means that If these conditions are maintained, then, there exists a relationship between class Q^* and square normal operators that need to be established.

The relationship between class Q^* with normal is given in Corollary 2.3.4 below.

Corollary 2.3.4: Wanjala and Nyongesa, (2021)
Let $T \in B(H)$, if $T \in Q^*$ such that T^2 is normal, then T is normal.

If T^2 is normal, then T is 2-normal.
This is an application of Jibril (2010) proposition 22 on class Q operators which shows that If $T \in B(H)$ is both 2-normal and in Q, then T is normal.

The adjoint of an operator preserves the inner product between vectors. Wanjala and Nyongesa (2021) studied the adjoint of operator T in class Q* operators and made the following observation.

Proposition 2.3.5: Wanjala and Nyongesa, (2021)
Let $T \in Q^*$, then $(TT^*)^2 = T^{*2}T^2$

Since $T \in Q^*$, then $T^* \in Q^*$.
Therefore,

$$\begin{aligned} ((T^*)^*(T^*))^2 &= (T^*)^2(T^*)^{*2} \\ &= (T^*)^2T^2 \\ &= (TT^*)^2 \\ &= T^{*2}T^2 \end{aligned}$$

This concept of the adjoint of an operator is applied in proposition 2.3.6 as follows.

Proposition 2.3.6: Wanjala and Nyongesa, (2021)
Let $T \in B(H)$, if T and T^{-1} are quasinormal, then $T \in Q^*$.
T being quasinormal implies

$$TT^*T = T^*T^2 \tag{2.5}$$

T^* being quasinormal implies

$$T^*TT^* = TT^{*2} \tag{2.6}$$

From (2.5) and (2.6) and getting the adjoint of (2.6) we have,

$$(T^*TT^* = TT^{*2})^* = TT^*T = T^*T^2 = T^2T^* \tag{2.7}$$

Post multiplying (2.7) by T^* , we have,

$$TT^*TT^* = TTT^*T^* = T^2T^*T^* = T^{*2}T^2 = T^2T^{*2} = (TT^*)^2$$

An extension of the adjoint of an operators to square normal operators was studied. According to Alzuraiqi and Patel (2010), the class of 2-normal operators is an extension of normal operators. In Jibril (2010), it was demonstrated that there exists an operator in Q that is not 2-normal. Wanjala and Nyongesa have also observed that a 2-normal operator may not necessarily belong to class Q^*. However, they have stated the relationship between quasinormal operators and Class Q^* in proposition 2.3.6.

Lemma 2.3.7: Wanjala and Nyongesa, (2021)
Let $T \in Q^*$. Then T is a square normal operator.

This can be proved directly from the definition of a square normal operator and then applying the norm of the operator as given in Corollary 2.2.2. However, this study established the relation between class Q^* operators are square normal.
This study also investigated some of the relationships of the class Q^*, Square normal operators with other classes of operators. These classes of operators are 2-normal operators and 3-normal operators.

CHAPTER THREE

RESEARCH METHODOLOGY

3.1 Study Site

This research was based at Chuka University Postgraduate Library.

3.2 General Approach

For successful research and completion of this study, background knowledge and an in-depth understanding of operators in Hilbert space were crucial. Knowledge of functional analysis and general topology was also very useful. In this research, known results of theorems without proof were adopted. The definition of terms and inclusions of operators was also adopted. Where possible, the construction of operator inclusions was done.

3.3 Technical Approach

Specifically, to achieve the objectives of this study, the approach applied in investigating the characterization and relations of Square normal and Class Q^* operators followed a methodical pathway of other operators in Hilbert spaces.

3.3.1 Characterization of Square Normal Operators

The following theorems were important in investigating the scalar addition and multiplication, sum and product and commutation relations to Square normal operators.

Theorem 3.3.1.1: Kreyszig, (1989)

Suppose we have two Hilbert spaces H_1, H_2 , with two bounded linear operators S and T such that $S : H_1 \to H_2$ and $T : H_1 \to H_2$. If α is any scalar. Then,

(i) $\langle T^*y, x\rangle = \langle y, Tx\rangle \forall x \in H_1, y \in H_2$

(ii) $(S + T)^* = S^* + T^*$

(iii) $(\alpha T)^* = \overline{\alpha} T^*$

(iv) $(T^*)^* = T$

(v) $\|T^*T\| = \|T^*\| = \|T\|^2$

(vi) $T^*T = 0 \iff T = 0$

(vii) $(ST)^* = T^*S^*$ assuming $H_2 = H_1$

Theorem 3.3.1.2: Beck and Putnam, (1962)

Let T be a bounded linear operator. The following are equivalent.

(i) T is normal.

(ii) T^* is normal.

(iii) The selfadjoint and anti-selfadjoint parts of T i.e.$T = T_1 + iT_2$, with $T_1 = \frac{T+T^*}{2}$ and $iT_2 = \frac{T-T^*}{2}$ commutes.

Theorem 3.3.1.3: Kreyszig, (1989)

Let T and S be two normal operators. If T commutes with S, then $T + S$ is normal. Indeed the proof follows by using the Fuglede theorem since the commutativity of T and S implies the commutativity of T and S^*. The converse of this theorem is not always true.

Theorem 3.3.1.4: Patel and Ramanujan, (1981)

The set of all normal operators on H is a closed subset of $B(H)$ which contains the set of all self-adjoint operators, and is closed under scalar multiplication.

Theorem 3.3.1.5: Alzuraiqi and Patel, (2010)

If T is a normal operator, then so are

(i) kT for any real number k.

(ii) any $S \in B(H)$ that is unitarily equivalent to T.

(iii) the restriction T/M of T to any closed subspace M of H that reduces T.

3.3.2 Characterization of Class Q^* Operators

As mentioned previously, Class Q operators differ from Class Q* operators, which means not all the characteristics of Class Q apply to this class. However, this study established some of the properties of Class Q that do apply to Class Q* operators. Furthermore, the following theorems were used to establish the additional properties on class Q^* operators.

Theorem 3.3.2.1: Fuglede, (1950)

If T and S are in $B(H)$ such that $TS = ST$ where S is normal,then $TS^* = S^*T$

Theorem 3.3.2.2: Putnam, (1951)

Let T and S be normal operators and X be an operator such that $TX = XS$.
Then $T^*X = XS^*$.

Theorem 3.3.2.3: Krutan and Luigj, (2013)

Let $T \in B(H)$ be in n-class Q operator. If T doubly commutes with an Isometric operator S, then TS is an operator of n-class Q.

Theorem 3.3.2.4: Panayappan and Sivamani, (2012)

If $T \in$ n-class Q operator, so are

(i) kT for any real number k

(ii) The restriction T/M of any closed subspace M of H that reduces T.

Theorem 3.3.2.5: Alzuraiqi and Patel, (2010).

Let S and T be two commuting n-normal operators. The product ST is also an n-normal operator.

If the operators S and T are non-commuting, then the above theorem is not necessarily true. The sum $S + T$ need not be normal.

3.3.3 Commutation Relations of Square Normal, Class Q^* and other Operators

It is noted that all normal operators are Square normal, and all class Q^* operators are Square normal operators. However, there is the existence of non-normal Square normal operators and therefore the relation between normal operators with class Q^* can not be generalized. To investigate the relationship between n-normal and square normal, normal operators and class Q^*, the following definitions were used.

Definition 3.3.3.1: Isometry operator. (Messaoud and Mostefa, 2016)

An operator $T \in B(H)$ is called isometry if $\|Tx\| = \|x\| \; \forall x \in H$ or equivalently $T^*T = I$.

Definition 3.3.3.2: Unitary operator. (Messaoud and Mostefa, 2016)

An operator $T \in B(H)$ is called unitary if and only if $TT^* = T^*T = I$

Trivially, every unitary operator is normal.

It is also worth noting that T is unitary if and only if it is invertible and $T^{-1} = T^*$.

Definition 3.3.3.3: n-normal operator. (Alzuraiqi and Patel, 2010)

An operator $T \in B(H)$ is said to be n-normal if $T^nT^* = T^*T^n$

3.4 Ethical Consideration

The proposal of this study was approved by Chuka University's ethical review committee and the research permit was obtained from the National Commission for Science, Technology, and Innovation (NACOSTI) before commencing the research. Ethical issues such as plagiarism, falsifying of the work, and biases was strictly observed. The outcomes of the study was carefully and critically reviewed so that the results obtained were credible and reported with honesty and integrity. The results were published transparently to help further knowledge advancement in areas of pure mathematics. Finally, the results from this study upon publishing will not be retrieved or transmitted without the knowledge of Chuka University.

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 Introduction

This chapter presents the findings of a study on the characterizations of square normal operators and the class Q^*. Additionally, the relationship between square normal operators, class Q^* operators and other operators on Hilbert spaces will be discussed.

4.2 Square Normal Operators

Mahmood (2016), extended the class of normal operators to a class of square normal operators and established that every normal operator is a square normal operator, but the reverse is not necessarily true. This study extends the properties of normal operators and their related operators to square normal operators. These properties include adjoint, inverse, unitary equivalence, scalar addition and multiplication, as well as the product and sum of operators. Many authors have investigated these properties on other operators in Hilbert spaces. For instance, Panayappan *et.al* (2012), investigated the above properties on n-binomial operators. Jibril (2013) investigated the concept of unitary equivalence and noted that any operator S unitarily equivalent to operator T in μ is also in μ. Devika and Suresh (2013) proved that if $T \in B(H)$ is of quasi-class Q, then T^{-1}, if it exists, is of quasi-class Q. Messaoud and Mostefa (2016) showed that any operator unitarily equivalent to an n-hyponormal operator is also an n-hyponormal operator. This concept of unitary equivalence is also extended to n-class Q operators. Also, according to Senhthilkumar and Parvatham (2017), for any T in n-class Q whose T^{-1} exists, T^{-1} is an operator of n-class Q. Wanjala and Nyongesa (2021) also showed that the above three properties hold for class Q^* operators.

The concept of scalar multiplication and scalar addition on operators in the Hilbert space has been investigated by many authors. Alzuraiqi and Patel (2010) established that n-normal operators are closed under scalar multiplication. According to Jibril (2011), operators whose squares are 2-normal are closed under scalar multiplication. However, Jibril (2011) also noted that the concept of scalar addition does not hold for (SN) operators. That is, if $T \in (SN)$, then it is not necessary that $T+\lambda I$ is in (SN). According to Panayappan *et.al* (2012), n-power class Q operators are closed under scalar multiplication for any scalar $\lambda \in \mathbb{R}$. Senhthilkumar and Parvatham (2017) noted that if $T \in B(H)$ is an n-class Q operator, then λT is also an n-class Q operator.

Conway (1985) studied the sum and product of two normal operators that commute with each other and obtained the following results.

Lemma 4.2.1: Conway, (1985)

If T and S are normal operators on H with the property that each commutes with the adjoint of the other, then

(i) $T + S$ is normal.

(ii) TS is normal.

In Alzuraiqi and Patel (2010), the authors extended this concept to n-normal operators and noted that the commutativity condition of n-normal operators should not be ignored as seen in lemma 4.2.2.

Lemma 4.2.2: Alzuraiqi and Patel, (2010)

If S and T are commuting n-normal operators, then ST is an n-normal operator.

However, if operators S and T are non-commuting, then ST are not necessarily n-normal. According to Alzuraiqi and Patel (2010), the sum of two commuting n-normal operators need not be n-normal as seen in the following example.

Given operators T and S as,

$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we have

$$TS = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$ST = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Now

$$S+T=\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}1&1\\0&1\end{bmatrix}$$

and

$$(S+T)^2=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}=\begin{bmatrix}1&2\\0&1\end{bmatrix}$$

$\begin{bmatrix}1&2\\0&1\end{bmatrix}$ is not normal and hence S+T is not an n-normal operator.

Jibril (2011) studied the sum and product of operators in SN and noted that the product of two operators in SN is also in SN provided that the operators commute with each other. However, the sum of two operators in SN is not necessarily in SN regardless of the commutation property of the two operators being satisfied. According to Panayappan *et.al* (2012), the product of two non-commuting 2-normal operators is not 2-normal and also proved that the sum and difference of two commuting n-normal operators need not be n-normal. Jibril (2013) states that μ operators are not closed under addition or multiplication. Messaoud and Mostefa (2016) noted that if T and S are 2-power-hyponormal operators, such that $TS^* = S^*T$ and $ST + TS = 0$, then $T + S$ and ST are 2-power-hyponormal.

4.2.1 Characterization of Square Normal Operators

Proposition 4.2.1.1 provides the results of algebraic properties: inverse, adjoint, and unitary equivalence of square normal operators in Hilbert spaces.

Proposition 4.2.1.1.
If an operator $T \in B(H)$ is square normal, then so are

(i) T^*.

(ii) T^{-1} if it exist.

(iii) Any operator $S \in B(H)$ that is unitarily equivalent to T.

Proof **(i)**
Since T is a square normal operator, we have

$$T^2T^{*2} = T^{*2}T^2 \tag{4.1}$$

Replacing T with T^* on the left of equation (4.1) we have

$$(T^*)^2(T^*)^{*2} = T^{*2}T^{**2} = T^{*2}T^2 \tag{4.2}$$

Replacing T with T^* on the right of equation (4.2) we have

$$(T^*)^{*2}(T^*)^2 = T^{**2}T^{*2} = T^2T^{*2} \tag{4.3}$$

Comparing equations (4.2) and (4.3), T^* is square normal.

Proof **(ii)**

$$\begin{aligned}(T^{-1})^2(T^{-1})^{*2} &= (T^2)^{-1}(T^{*2})^{-1} \\ &= (T^2T^{*2})^{-1} \\ &= (T^{*2})^{-1}(T^2)^{-1} \\ &= (T^{-1})^{*2}(T^{-1})^2\end{aligned}$$

Hence T^{-1} is square normal.

Proof **(iii)**

Let $S \in B(H)$ which is unitarily equivalent to T. There is a unitary operator $U \in B(H)$ such that $S = U^*TU$ which implies that $S^* = (U^*TU)^* = U^*T^*U$.

$$\begin{aligned}S^2S^{*2} &= U^*TUU^*TUU^*T^*UU^*T^*U \quad (UU^* = I) \\ &= U^*T^2T^{*2}U\end{aligned} \tag{4.4}$$

$$\begin{aligned}S^{*2}S^2 &= U^*T^*UU^*T^*UU^*TUU^*TU \\ &= U^*T^*T^*TTU \\ &= U^*T^{*2}T^2U\end{aligned} \tag{4.5}$$

Since T is square normal, we have $U^*T^2T^{*2}U = U^*T^{*2}T^2U$. This implies that

$S^2S^{*2} = S^{*2}S^2$ and so S is a square normal operator. □

The concept of the closure of an operator is crucial in mathematics as it allows us to extend the operator to another well-defined operator. Theorem 4.2.1.2 demonstrates that square normal operators exhibit closure under scalar multiplication and addition.

Theorem 4.2.1.2.
Let $T \in B(H)$ be a square normal operator. Then, if T is a normal operator, for any scalar $\lambda \in (\mathbb{C})$,

(i) λT is square normal.

(ii) $T + \lambda$ is square normal.

***Proof* (i)**

$$\begin{aligned}
(\lambda T)^2(\lambda T)^{*2} &= (\lambda)^2(T)^2(\lambda^*)^2(T^*)^2 \\
&= (\lambda)^2(\lambda^*)^2(T)^2(T^*)^2 \\
&= (\lambda^*)^2(\lambda)^2(TT^*)^2 \\
&= (\lambda^*\lambda)^2(T^*T)^2 \\
&= (\lambda^*)^2(\lambda)^2(T^*)^2(T)^2 \\
&= (\lambda^*)(T^*)^2(\lambda)^2(T)^2 \\
&= (\lambda^*T^*)^2(\lambda)^2(T)^2 \\
&= (\lambda T)^{*2}(\lambda T)^2
\end{aligned}$$

Hence (λT) is square normal.

***Proof* (ii)**

Suppose on the contrary $(T + \lambda)$ is not square normal.Then,

$$(T + \lambda)^2(T + \lambda)^{*2} - (T + \lambda)^{*2}(T + \lambda)^2 \neq 0 \tag{4.6}$$

Then

$$T^2T^{*2} + \lambda TT^{*2} - T^{*2}T^2 - \lambda T^{*2}T \neq 0 \tag{4.7}$$

Since T is normal, then T is square normal and therefore,

$$T^2T^{*2} = T^{*2}T^2 \tag{4.8}$$

Equation (4.7) becomes,

$$\lambda TT^{*2} - \lambda T^{*2}T \neq 0 \tag{4.9}$$

Multiplying equation (4.9) with T^* from left on both sides we get

$$\lambda T^*TT^{*2} - \lambda T^*T^{*2}T \neq 0 \tag{4.10}$$

Multiply equation (4.10) with T from right on both sides we get

$$\lambda T^*TT^{*2}T - \lambda T^*T^{*2}T^2 \neq 0 \tag{4.11}$$

Since T is normal, equation (4.11) becomes

$$\lambda T^*TT^*T^*T - \lambda T^*T^{*2}T^2 \neq 0 \tag{4.12}$$

$$\lambda T^*T^*TT^*T - \lambda T^*T^{*2}T^2 \neq 0 \tag{4.13}$$

Equation (4.13) yields

$$\lambda T^*T^*T^*TT - \lambda T^*T^{*2}T^2 \neq 0 \tag{4.14}$$

By simplifying equation (4.14) we obtain

$$\lambda T^*T^{*2}T^2 - \lambda T^*T^{*2}T^2 \neq 0 \tag{4.15}$$

A contradiction, therefore

$$\lambda T^*T^{*2}T^2 = \lambda T^*T^{*2}T^2 \tag{4.16}$$

This implies that the assumption that $T + \lambda$ is not square normal was wrong. So $T + \lambda$ is square normal and therefore

$$(T+\lambda)^2(T+\lambda)^{*2} = (T+\lambda)^{*2}(T+\lambda)^2$$

$\square$

Note that the condition T is normal should not be ignored otherwise $T+\lambda$ will not be square normal.

Example 4.2.1.3 illustrates the existence of a square normal operator that is not normal as seen in mahmood, (2016). As noted, scalar addition does not hold for such operators.

Example 4.2.1.3: Mahmood, (2016)

Let T be given by $T = \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix}$ where $i \in \mathbb{C}$, then $T^* = \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix}$

We have

$$TT^* = \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix} \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$T^*T = \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Since $TT^* \neq T^*T$, T is not normal.

$$T^2 = \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix} \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T^{*2} = \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T^2T^{*2} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T^{*2}T^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$T^2T^{*2} = T^{*2}T^2$ and hence square normal.

Therefore, T is square normal but not normal.

Let $\lambda = i$,$\lambda I = \begin{bmatrix} i & o \\ o & i \end{bmatrix}$

Then

$$T + \lambda I = \begin{bmatrix} 2i & 0 \\ i & 0 \end{bmatrix}$$

$$(T + \lambda I)^2 = \begin{bmatrix} -4 & 0 \\ -2 & 0 \end{bmatrix}$$

$$(T + \lambda I)^* = \begin{bmatrix} -2i & -i \\ 0 & 0 \end{bmatrix}$$

$$(T + \lambda I)^{*2} = \begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix}$$

$$(T + \lambda I)^2(T + \lambda I)^{*2} = \begin{bmatrix} -4 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 8 & 4 \end{bmatrix}$$

$$(T+\lambda I)^{*2}(T+\lambda I)^2 = \begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 20 & 0 \\ 8 & 0 \end{bmatrix}$$

Hence $T+\lambda I$ is not a square normal.

The theorem 4.2.1.4 shows that the sum and product of two commuting square normal operators is a square normal operator.

Theorem 4.2.1.4.
Let H be a Hilbert space and T and S be normal operators on $B(H)$. If $(T+S)(T+S)^* = (T+S)^*(T+S)$ and $(TS)(TS)^* = (TS)^*(TS)$, then

(i) $T+S$ is square normal.

(ii) TS is square normal

Proof **(i)**

$$\begin{aligned}
(T+S)^2(T+S)^{*2} &= (T+S)(T+S)(T^*+S^*)(T^*+S^*) \\
&= (T+S)(T^*+S^*)(T+S)(T^*+S^*) \\
&= (T^*+S^*)(T+S)(T^*+S^*)(T+S) \\
&= (T^*+S^*)(T^*+S^*)(T+S)(T+S) \\
&= (T+S)^*(T+S)^*(T+S)(T+S) \\
&= (T+S)^{*2}(T+S)^2
\end{aligned}$$

Hence square normal.

Proof **(ii)**

$$\begin{aligned}(TS)^2(TS)^{*2} &= (TS)(TS)(TS)^*(TS)^* \\ &= (TS)(TS)^*(TS)(TS)^* \\ &= (TS)^*(TS)(TS)^*(TS) \\ &= (TS)^*(TS)^*(TS)(TS) \\ &= (TS)^{*2}(TS)^2\end{aligned}$$

Hence square normal. □

Example 4.2.1.5

Let $T = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ and $S = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$ be two normal operators.

Then we have the following

$$T + S = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} + \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix} = \begin{bmatrix} i+1 & i \\ -i & i+2 \end{bmatrix}$$

$$(T+S)^* = \begin{bmatrix} -i+1 & i \\ -i & -i+2 \end{bmatrix}$$

$$(T+S)(T+S)^* = \begin{bmatrix} i+1 & i \\ -i & i+2 \end{bmatrix} \begin{bmatrix} -i+1 & i \\ -i & -i+2 \end{bmatrix} = \begin{bmatrix} 3 & 3i \\ -3i & 6 \end{bmatrix}$$

$$(T+S)^*(T+S) = \begin{bmatrix} -i+1 & i \\ -i & -i+2 \end{bmatrix} \begin{bmatrix} i+1 & i \\ -i & i+2 \end{bmatrix} = \begin{bmatrix} 3 & 3i \\ -3i & 6 \end{bmatrix}$$

Therefore $(T+S)(T+S)^* = (T+S)^*(T+S)$

Now, we show that $T+S$ is Square normal,

$$(T+S)^{*2} = \begin{bmatrix} -i+1 & i \\ -i & -i+2 \end{bmatrix} \begin{bmatrix} -i+1 & i \\ -i & -i+2 \end{bmatrix} = \begin{bmatrix} -2i+1 & 2+3i \\ -2-3i & -4i+4 \end{bmatrix}$$

$$(T+S)^{2} = \begin{bmatrix} i+1 & i \\ -i & i+2 \end{bmatrix} \begin{bmatrix} i+1 & i \\ -i & i+2 \end{bmatrix} = \begin{bmatrix} 2i+1 & -2+3i \\ 2-3i & 4i+4 \end{bmatrix}$$

$$(T+S)^{2}(T+S)^{*2} = \begin{bmatrix} 2i+1 & -2+3i \\ 2-3i & 4i+4 \end{bmatrix} \begin{bmatrix} -2i+1 & 2+3i \\ -2-3i & -4i+4 \end{bmatrix} = \begin{bmatrix} 18 & 27i \\ -27i & 45 \end{bmatrix}$$

$$(T+S)^{*2}(T+S)^{2} = \begin{bmatrix} -2i+1 & 2+3i \\ -2-3i & -4i+4 \end{bmatrix} \begin{bmatrix} 2i+1 & -2+3i \\ 2-3i & 4i+4 \end{bmatrix} = \begin{bmatrix} 18 & 27i \\ -27i & 45 \end{bmatrix}$$

Therefore, $(T+S)^2(T+S)^{*2} = (T+S)^{*2}(T+S)^2$ and hence $T+S$ is Square normal.

$$TS = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix} = \begin{bmatrix} i & -1 \\ 1 & 2i \end{bmatrix}$$

$$(TS)^{*} = \begin{bmatrix} -i & 1 \\ -1 & -2i \end{bmatrix}$$

$$(TS)(TS)^{*} = \begin{bmatrix} i & -1 \\ 1 & 2i \end{bmatrix} \begin{bmatrix} -i & 1 \\ -1 & -2i \end{bmatrix} = \begin{bmatrix} 2 & 3i \\ -3i & 5 \end{bmatrix}$$

and

$$(TS)^{*}(TS) = \begin{bmatrix} -i & 1 \\ -1 & -2i \end{bmatrix} \begin{bmatrix} i & -1 \\ 1 & 2i \end{bmatrix} = \begin{bmatrix} 2 & 3i \\ -3i & 5 \end{bmatrix}$$

Consequently,

$$(TS)^2 = \begin{bmatrix} i & -1 \\ 1 & 2i \end{bmatrix} \begin{bmatrix} i & -1 \\ 1 & 2i \end{bmatrix} = \begin{bmatrix} -2 & -3i \\ 3i & -5 \end{bmatrix}$$

$$(TS)^{*2} = \begin{bmatrix} -i & 1 \\ -1 & -2i \end{bmatrix} \begin{bmatrix} -i & 1 \\ -1 & -2i \end{bmatrix} = \begin{bmatrix} -2 & -3i \\ 3i & -5 \end{bmatrix}$$

$$(TS)^2(TS)^{*2} = \begin{bmatrix} -2 & -3i \\ 3i & -5 \end{bmatrix} \begin{bmatrix} -2 & -3i \\ 3i & -5 \end{bmatrix} = \begin{bmatrix} 13 & 21i \\ -21i & 34 \end{bmatrix} = (TS)^{*2}(TS)^2$$

Hence TS is square normal.

Next, we give an example to show that if two square normal operators T and S do not commute, then $S + T$ is not necessarily a square normal operator.

Example 4.2.1.6

Let two operators T and S be two operators in $\mathbb{R}^2$.

$$T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, S = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Then,

$$T^* = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, S^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$TT^* = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$T^*T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since T is normal, it is square normal.

On the other hand, S=S^*. This is a Hermitian operator which is normal implying square normal.

$$TS = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$ST = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Clearly, $TS \neq ST$ hence non-commutative.

We now have,

$$T + S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

and

$$(T + S)^2 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$(T + S)^* = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

and

$$(T + S)^{*2} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$(T + S)^2(T + S)^{*2} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$(T + S)^{*2}(T + S)^2 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

This shows that $(T+S)^2(T+S)^{*2} \neq (T+S)^{*2}(T+S)^2$ and therefore $T+S$ is not square normal.

4.3 Class Q^* Operators

Jibril (2010) introduced class Q operators on a Hilbert space where an operator $T \in B(H)$ is in class Q if $T^{*2}T^2 = (T^*T)^2$. Class Q operators were enlarged to n power class Q operators by Panayappan and Sivamani (2012). Wanjala and Nyongesa (2021) studied class Q^* operators, showed that class Q^* operators differ from class Q operators, and stated some of the properties as seen in Proposition 4.3.1.

Proposition 4.3.1: Wanjala and Nyongesa, (2021)

Let $T \in B(H)$, if $T \in Q^*$, then the following hold:

(i) T^{*2} is in Q^*

(ii) $T^{-1} \in Q^*$ provided it exists.

(iii) Any operator $S \in B(H)$ that is unitarily equivalent to T is also in Q^*.

This section will focus on various properties of class Q^* operators that have not yet been studied. These properties include Cartesian decomposition, sum and product, direct sum and the tensor product of class Q^* operators.

Example 4.3.2 illustrates that class Q^* operators are not convex.

Example 4.3.2

Consider two operators $T, S \in B(H)$ such that $T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

Then,

$$T^* = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

$$T^{*2} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$T^{*2}T^2 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$TT^* = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(TT^*)^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

Hence $T \in Q^*$

$$S^* = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S^2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S^{*2} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S^{*2}S^2 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$$

$$SS^* = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(SS^*)^2 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore $S \in Q^*$.

Let $M = \frac{1}{2}T + \frac{1}{2}S$

$$M = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{5}{4} \\ \frac{5}{4} & \frac{3}{4} \end{bmatrix}$$

$$M^* = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

$$M^{*2} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 2 & \frac{5}{4} \\ -\frac{5}{4} & \frac{3}{4} \end{bmatrix}$$

$$M^{*2}M^2 = \begin{bmatrix} 2 & \frac{5}{4} \\ -\frac{5}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 2 & -\frac{5}{4} \\ \frac{5}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{89}{16} & -\frac{25}{16} \\ -\frac{25}{16} & \frac{34}{16} \end{bmatrix}$$

$$MM^* = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{10}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{5}{4} \end{bmatrix}$$

$$(MM^*)^2 = \begin{bmatrix} \frac{10}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{5}{4} \end{bmatrix} \begin{bmatrix} \frac{10}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{5}{4} \end{bmatrix} = \begin{bmatrix} \frac{101}{16} & \frac{45}{16} \\ \frac{15}{16} & \frac{26}{16} \end{bmatrix}$$

Now $M^{*2}M^2 \neq (MM^*)^2$, $M \notin Q^*$ and therefore Q^* is not convex.

4.3.1 Characterization of Class Q^* Operators

Many authors have investigated the property of Cartesian decomposition on different operators in Hilbert spaces. Alzuraiqi and Patel (2010) studied Cartesian decomposition on n-normal operators and the results are stated in Proposition 4.3.1.1

Proposition 4.3.1.1: Alzuraiqi and Patel, (2010)

Let $T \in B(H)$ with the Cartesian decomposition $T = A + iB$ where A and B are self-adjoint operators. Then T is a 2-normal operator if and only if B^2 commutes with A and A^2 commutes with B.

Other authors such as Panayappan and Sivamani (2012) proved that if $T \in B(H)$ with the Cartesian decomposition $T = A + iB$ where A and B are self adjoint operators, then T is binormal if and only if

(i) $AB^3 + B^3A = A^3B + BA^3$

(ii) $A^2BA + ABA^2 = B^2AB + BAB^2$

Jibril (2013) showed that if $T \in B(H)$ with the Cartesian decomposition $T = A + iB$, then $T \in \mu$ if and only if $A^2 = B^2$.

Theorem 4.3.1.2 gives the results for the Cartesian decomposition on class Q^* operators on a Hilbert space.

Theorem 4.3.1.2

Let $T \in B(H)$ with the Cartesian decomposition $T = A + iB$ where A and B are self-adjoint operators. Then $T \in Q^*$ if

(i) $AB = BA$

(ii) $3A^3B - 2B^3A - 2BA^3 - ABA^2 = 0$

Proof

Since $T \in Q^*$, then $T^{*2}T^2 = (TT^*)^2$

Now $T = A + iB$, $T^* = A - iB$

$$TT^* = (A + iB)(A - iB) = A^2 - iAB + iBA + B^2$$

$$T^{*2} = (A - iB)(A - iB) = A^2 - iAB - iBA - B^2$$

$$T^2 = (A + iB)(A + iB) = A^2 + iAB + iBA - B^2$$

$$\begin{aligned} T^{*2}T^2 &= (A^2 - iAB - iBA - B^2)(A^2 + iAB + iBA - B^2) \\ &= A^4 + iA^3B + iA^2BA - A^2B^2 - iABA^2 + ABAB \\ &\quad + AB^2A - iAB^3 - iBA^3 + BA^2B + BABA \\ &\quad + iBAB^2 - B^2A^2 - iB^2AB - iB^3A + B^4 \end{aligned} \tag{4.17}$$

$$\begin{aligned} (TT^*)^2 &= (A^2 - iAB + iBA + B^2)(A^2 - iAB + iBA + B^2) \\ &= A^4 - iA^3B + iA^2BA + A^2B^2 - iA^3B - ABAB \\ &\quad + AB^2A - iAB^3 + iBA^3 + BA^2B - BABA \\ &\quad + iBAB^2 + B^2A^2 - iB^2AB + iB^3A + B^4 \end{aligned} \tag{4.18}$$

Since $T \in Q^*$, then equation (4.17) must equate to equation (4.18).
On further simplification,

$$\begin{aligned} T^{*2}T^2 &= (TT^*)^2 \\ &= iA^3B - A^2B^2 - iABA^2 + ABAB - iAB^3 + BA^2B + BABA - B^2A^2 - iB^3A \\ &= -iA^3B + A^2B^2 - iA^3B - ABAB + iBA^3 - BABA + B^2A^2 + iB^3A \end{aligned}$$

Equating the real part, we have

$$ABAB + BABA - A^2B^2 = A^2B^2 + B^2A^2 - ABAB - BABA$$

$$ABAB + BABA = A^2B^2 + B^2A^2$$

This is true if A commutes with B. Thus,

$$AB = BA$$

Equating the imaginary part, we have

$$iA^3B - iABA^2 - iBA^3 - iB^3A = -iA^3B - iA^3B + iBA^3 + iB^3A$$

$$A^3B - ABA^2 - BA^3 - B^3A = -A^3B - A^3B + BA^3 + B^3A$$

Which means

$$3A^3B - 2B^3A - 2BA^3 - ABA^2 = 0$$

$\square$

Theorem 4.3.1.3 states result on the commutation relation in class Q^* operators.

Theorem 4.3.1.3

Let $T$and S be bounded linear operators in Q^* such that the sum $(T + S)$ commutes with $(T + S)^*$. Then $(T + S) \in Q^*$.

Proof

$$\begin{aligned}
(T + S)^{*2}(T + S)^2 &= (T + S)^*(T + S)^*(T + S)(T + S) \\
&= (T + S)^*(T + S)(T + S)^*(T + S) \\
&= (T + S)(T + S)^*(T + S)(T + S)^* \\
&= (T + S)(T + S)(T + S)^*(T + S)^* \\
&= (T + S)^2(T + S)^{*2} \\
&= ((T + S)(T + S)^*)^2
\end{aligned}$$

Hence $(T + S) \in Q^*$

$\square$

Example 4.3.1.4

Let $T, S \in B(H)$ such that $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

Then, $T, S \in Q^*$

Now,

$$(T+S) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(T+S)^* = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(T+S)(T+S)^* = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

$$(T+S)^*(T+S) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

Now

$$(T+S)^{*2} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

$$(T+S)^2 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

$$(T+S)^{*2}(T+S)^2 = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 81 & 0 \\ 0 & 16 \end{bmatrix}$$

$$(TT^*)^2 = (\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix})^2 = (\begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix})^2 = \begin{bmatrix} 81 & 0 \\ 0 & 16 \end{bmatrix}$$

Hence $(T + S) \in Q^*$

Remark 4.3.1.5

If two operators $T, S \in Q^*$ are such that the sum $T + S$ does not commute with $(T + S)^*$ then $T + S$ is not necessarily in class Q^* operators.

Example 4.3.1.6 shows that the sum of two operators T and S in class Q^* such that $(T + S)(T + S)^* \neq (T + S)^*(T + S)$ does not belong to class Q^*.

Example 4.3.1.6

Let $T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ be two operators in $B(H)$.

Then,

$T^* = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$T^2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

$$T^{*2} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$T^{*2}T^2 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$TT^* = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(TT^*)^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

Hence $T \in Q^*$

$$S^* = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S^2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S^{*2} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S^{*2}S^2 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$$

$$SS^* = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(SS^*)^2 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore $S \in Q^*$.

Now,

$$T + S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$$

$$(T + S)^* = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$(T+S)(T+S)^* = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ 1 & 5 \end{bmatrix}$$

$$(T+S)^*(T+S) = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 10 & -1 \\ -1 & 5 \end{bmatrix}$$

Therefore $(T+S)$ is not normal.

Consequently,

$$(T+S)^2 = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & -5 \\ 5 & 3 \end{bmatrix}$$

$$(T+S)^{*2} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$(T+S)^{*2}(T+S)^2 = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 8 & -5 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 89 & -25 \\ -25 & 34 \end{bmatrix}$$

$$((T+S)(T+S)^*)^2 = (\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix})^2 = \begin{bmatrix} 10 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 10 & 1 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 102 & 15 \\ 15 & 6 \end{bmatrix}$$

Clearly $(T+S)^{*2}(T+S)^2 \neq ((T+S)(T+S)^*)^2$ and so$(T+S) \notin Q^*$

The result on the product of two class Q^* operators is given in theorem 4.3.1.7

Theorem 4.3.1.7

Let $T$and S be bounded linear operators in Q^* such that $TS^* = S^*T$ and $ST^* = T^*S$. Then $(TS) \in Q^*$.

Proof

$$
\begin{aligned}
(TS)^{*2}(TS)^2 &= (TS)^*(TS)^*(TS)(TS)\\
&= S^*T^*S^*T^*TSTS\\
&= S^*T^*S^*TT^*STS\\
&= T^*S^*TS^*ST^*TS\\
&= T^*TS^*SS^*TST^*\\
&= TT^*SS^*TS^*ST^*\\
&= TST^*TS^*SS^*T^*\\
&= TSTT^*SS^*T^*S^*\\
&= TSTST^*S^*T^*S^*\\
&= (TS)^2((T^*S^*))^2\\
&= (TS)^2((TS)^*))^2\\
&= ((TS)(TS)^*)^2
\end{aligned}
$$

Now $(TS)^{*2}(TS)^2 = ((TS)(TS)^*)^2$ implying that $(TS) \in Q^*$

□

Example 4.3.1.8

Consider two operators $T, S \in Q^*$ such that $T = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$T^* = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$ and $S^* = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$$TS^* = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} -2i & -2i \\ 2i & -2i \end{bmatrix}$$

$$S^*T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -2i & -2i \\ 2i & -2i \end{bmatrix}$$

Therefore $TS^* = S^*T$

$$T^*S = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 2i & -2i \\ 2i & 2i \end{bmatrix}$$

$$ST^* = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 2i & -2i \\ 2i & 2i \end{bmatrix}$$

Therefore $T^*S = ST^*$

$$TS = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} -2i & -2i \\ 2i & -2i \end{bmatrix}$$

Let the product TS be M $M$is a class Q^* operator since

$$M = \begin{bmatrix} -2i & -2i \\ 2i & -2i \end{bmatrix}, M^* = \begin{bmatrix} 2i & -2i \\ 2i & 2i \end{bmatrix}$$

$$M^{*2} = \begin{bmatrix} 2i & -2i \\ 2i & 2i \end{bmatrix} \begin{bmatrix} 2i & -2i \\ 2i & 2i \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ -8 & 0 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} -2i & -2i \\ 2i & -2i \end{bmatrix} \begin{bmatrix} -2i & -2i \\ 2i & -2i \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix}$$

$$M^{*2}M^2 = \begin{bmatrix} 0 & 8 \\ -8 & 0 \end{bmatrix} \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix} = \begin{bmatrix} 64 & 0 \\ 0 & 64 \end{bmatrix}$$

$$MM^* = \begin{bmatrix} -2i & -2i \\ 2i & -2i \end{bmatrix} \begin{bmatrix} 2i & -2i \\ 2i & 2i \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

$$(MM^*)^2 = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 64 & 0 \\ 0 & 64 \end{bmatrix}$$

Therefore M is class Q^* since $M^{*2}M^2 = (MM^*)^2$.

Remarks 4.3.1.9

The commutation relation of the operator with the adjoint of the other should not be ignored; otherwise, the operators' product will not be class Q^* as seen in example 4.3.1.10.

Example 4.3.1.10

Let $T, S \in B(H)$ such that $T = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$T^* = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$T^{*2} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$T^{*2}T^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$$

$$TT^* = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$(TT^*)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$$

Since $T^{*2}T^2 = (TT^*)^2$, $T \in Q^*$.

$$S^* = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$S^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$S^{*2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$S^{*2}S^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$SS^* = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$(SS^*)^2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Since $S^{*2}S^2 = (SS^*)^2$, $S \in Q^*$.

$$TS^* = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \neq S^*T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$$

$$T^*S = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \neq ST^* = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$$

Consequently, the product $TS \notin Q^*$.

Let $M = TS = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$

$$M^* = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$$

$$M^{*2} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$M^{*2}M^2 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$MM^* = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$(MM^*)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$$

Hence$M^{*2}M^2 \neq (MM^*)^2$.

The study of direct sum and tensor product of operators in Hilbert spaces has been a topic of interest for many researchers. Jibril (2010), proved that the direct sum and tensor product of two operators in SN are in SN. Later, Jibril (2013) showed that the direct sum and tensor product of two operators in μ are in μ. Messaoud and Mostefa (2016) showed that if $T_1, T_2, ...T_m$ are n-power-hypornormal operators in $B(H)$, then $T_1 \oplus T_2 \oplus ... \oplus T_m$ and $T_1 \otimes T_2 \otimes ... \otimes T_m$ are n-power-hyponormal operators.
Theorem 4.3.1.11 gives the results of the direct sum and tensor product of operators in class Q^*.

Theorem 4.3.1.11

Let $T_1, T_2, ...T_m$ be normal operators in class Q^*, Then;

(i) $T_1 \oplus T_2 \oplus ... \oplus T_m \in Q^*$

(ii) $T_1 \otimes T_2 \otimes ... \otimes T_m \in Q^*$

Proof. (i)

$(T_1 \oplus T_2 \oplus ... \oplus T_m)^{*2}(T_1 \oplus T_2 \oplus ... \oplus T_m)^2$
$= (T_1 \oplus T_2 \oplus ... \oplus T_m)^*(T_1 \oplus T_2 \oplus ... \oplus T_m)^*(T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1 \oplus T_2 \oplus ... \oplus T_m)$
$= (T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)(T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1 \oplus T_2 \oplus ... \oplus T_m)$
$= (T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)(T_1^*T_1 \oplus T_2^*T_2 \oplus ... \oplus T_m^*T_m)(T_1 \oplus T_2 \oplus ... \oplus T_m)$
$= (T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)(T_1T_1^* \oplus T_2T_2^* \oplus ... \oplus T_mT_m^*)(T_1 \oplus T_2 \oplus ... \oplus T_m)$
$= (T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)(T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)(T_1 \oplus T_2 \oplus ... \oplus T_m)$
$= (T_1^*T_1 \oplus T_2^*T_2 \oplus ... \oplus T_m^*T_m)(T_1^*T_1 \oplus T_2^*T_2 \oplus ... \oplus T_m^*T_m)$
$= (T_1T_1^* \oplus T_2T_2^* \oplus ... \oplus T_mT_m^*)(T_1T_1^* \oplus T_2T_2^* \oplus ... \oplus T_mT_m^*)$
$= (T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)(T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)$
$= (T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1^*T_1 \oplus T_2^*T_2 \oplus ... \oplus T_m^*T_m)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)$
$= (T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1T_1^* \oplus T_2T_2^* \oplus ... \oplus T_mT_m^*)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)$
$= (T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)$
$= (T_1 \oplus T_2 \oplus ... \oplus T_m)^2(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)^2$
$= ((T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*))^2$

Now, $(T_1 \oplus T_2 \oplus ... \oplus T_m)^{*2}(T_1 \oplus T_2 \oplus ... \oplus T_m)^2 = ((T_1 \oplus T_2 \oplus ... \oplus T_m)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*))^2$

Hence $T_1 \oplus T_2 \oplus ... \oplus T_m \in Q^*$

Proof. (ii)

Let $x_1, x_2, ...x_m \in H$

$(T_1 \otimes T_2 \otimes ... \otimes T_m)^{*2}(T_1 \otimes T_2 \otimes ... \otimes T_m)^2(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1 \otimes T_2 \otimes ... \otimes T_m)^*(T_1 \otimes T_2 \otimes ... \otimes T_m)^*(T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1 \otimes T_2 \otimes ... \otimes T_m)(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1 \otimes T_2 \otimes ... \otimes T_m)(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(T_1^*T_1 \otimes T_2^*T_2 \otimes ... \otimes T_m^*T_m)(T_1 \otimes T_2 \otimes ... \otimes T_m)(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(T_1T_1^* \otimes T_2T_2^* \otimes ... \otimes T_mT_m^*)(T_1 \otimes T_2 \otimes ... \otimes T_m)(x_1 \otimes x_2 \otimes ... \otimes x_m)$

$= (T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(T_1 \otimes T_2 \otimes ... \otimes T_m)(x_1 \otimes$
$x_2 \otimes ... \otimes x_m)$
$= (T_1^*T_1 \otimes T_2^*T_2 \otimes ... \otimes T_m^*T_m)(T_1^*T_1 \otimes T_2^*T_2 \otimes ... \otimes T_m^*T_m)(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1T_1^* \otimes T_2T_2^* \otimes ... \otimes T_mT_m^*)(T_1T_1^* \otimes T_2T_2^* \otimes ... \otimes T_mT_m^*)(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(x_1 \otimes$
$x_2 \otimes ... \otimes x_m)$
$= (T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1^*T_1 \otimes T_2^*T_2 \otimes ... \otimes T_m^*T_m)T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1T_1^* \otimes T_2T_2^* \otimes ... \otimes T_mT_m^*)T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= (T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)(x_1 \otimes$
$x_2 \otimes ... \otimes x_m)$
$= (T_1 \otimes T_2 \otimes ... \otimes T_m)^2(T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)^2(x_1 \otimes x_2 \otimes ... \otimes x_m)$
$= ((T_1 \otimes T_2 \otimes ... \otimes T_m)(T_1^* \otimes T_2^* \otimes ... \otimes T_m^*))^2(x_1 \otimes x_2 \otimes ... \otimes x_m)$
Therefore $T_1 \otimes T_2 \otimes ... \otimes T_m \in Q^*$ □

4.4 Relationship between Square Normal Operators and ClassQ^* Operators and other Operators on Hilbert Spaces

Authors such as Jibril (2011) have studied the connections between various operators in Hilbert spaces. According to Jibril (2011), if T, a class of operators whose squares are 2-normal is an isometry, then T is unitary. This has enabled the extensions of properties of one operator to another. Furthermore, the study has enabled authors to identify independent operators and also unravel properties that can be attached to one operator to establish a connection with another. For instance, Krutan and Luigj (2013) proved that if T, a bounded linear operator is n-power class Q and has an inverse, then T is an n-normal operator. Mahmood (2016) established the relationship between normal and square normal operators and proved that every normal operator is a square normal operator but the converse is not true. Furthermore, Mahmood (2016) showed that T is a square normal operator if and only if T^2 is normal. Proposition 4.4.1 and Proposition 4.4.2 shows the relationship between normal and square normal operators.

Proposition 4.4.1: Mahmood, (2016)
Let T be a normal operator. Then T is a square normal operator.

Proof
Since T is a normal operator,
Then,

$$TT^* = T^*T \tag{4.19}$$

Squaring both sides of equation (4.19), we get

$$(TT^*)^2 = (T^*T)^2 \tag{4.20}$$

Using the adjoint of an operator property in (4.20) to get (4.21)

$$T^2T^{*2} = T^{*2}T^2 \tag{4.21}$$

From (4.21), T is square normal.

Proposition 4.4.2: Mahmood, (2016)

T is square normal if and only if T^2 is normal.

Proof

Let T be a square normal operator and show that T^2 is normal.

Since T is square normal, then, by definition of square normal operators,

$$T^2(T^*)^2 = (T^*)^2T^2 \tag{4.22}$$

By the property of adjoint of an operator, equation (4.29) yields equation (4.30)

$$T^2(T^2)^* = (T^2)^*T^2 \tag{4.23}$$

This implies that T^2 is normal.

Conversely, suppose T^2 is normal and show T is square normal.

Since T^2 is normal, then, by definition of a normal operator, equation (4.31) is obtained.

$$T^2(T^2)^* = (T^2)^*T^2 \tag{4.24}$$

From the property of adjoint of an operator, $(T^2)^* = (T^*)^2$ and equation (4.31) yields

$$T^2(T^*)^2 = (T^*)^2T^2 \tag{4.25}$$

This implies that T is square normal. $\square$

Remark 4.4.3

If T is a square normal such that $T^2 = 0$, then it is not necessarily that $T = 0$.

Consider $T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ acting on $\mathbb{R}^2$.

Clearly, $T \neq 0$

But,

$$T^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Furthermore,

$$T^{*2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T^{*2}T^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T^2T^{*2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now , $T^{*2}T^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = T^2T^{*2}$

Hence T is square normal.

As seen in proposition 2.1.5, there exist square normal operators which are not normal.

T is not normal since

$$TT^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$T^*T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Theorem 4.4.4 gives the relationship between 2- normal and square normal operators.

Theorem 4.4.4

Let $T \in B(H)$ be a 2-normal operator. If T is normal, then $T \in B(H)$ is a square normal operator.

Proof

Since T is a 2-normal operator, from the definition,

$$T^*T^2 = T^2T^* \tag{4.26}$$

Multiply equation (4.26) by T^* from the left to obtain

$$T^*T^*T^2 = T^*T^2T^* \tag{4.27}$$

$$\begin{aligned} T^*T^*T^2 = (T^*)^2T^2 &= T^*TTT^* \\ &= TT^*TT^* (\text{using normality of } T) \\ &= TTT^*T^* (\text{using normality of } T) \\ &= T^2(T^*)^2 \end{aligned}$$

Hence T is square normal. □

Authors such as Panayappan and Sivamani (2012) have studied the independence of operators on a Hilbert space. According to Panayappan and Sivamani (2012), the class of 2 power class Q need not be 3 power class Q and vice versa. Example 4.4.5 and Example 4.4.6 show that 2 power class Q operators and 3 power class Q operators are independent.

Example 4.4.5

Consider an operator $T = \begin{bmatrix} i & 2 \\ 0 & -i \end{bmatrix}$ acting on a 2-dimensional Hilbert space.

$$T^* = \begin{bmatrix} -i & 0 \\ 2 & i \end{bmatrix}$$

$$T^2 = \begin{bmatrix} i & 2 \\ 0 & -i \end{bmatrix} \begin{bmatrix} i & 2 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T^{*2} = \begin{bmatrix} -i & 0 \\ 2 & i \end{bmatrix} \begin{bmatrix} -i & 0 \\ 2 & i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T^4 = T^2T^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T^{*2}T^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T^*T^2 = \begin{bmatrix} -i & 0 \\ 2 & i \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} i & 0 \\ -2 & -i \end{bmatrix}$$

$$(T^*T^2)^2 = \begin{bmatrix} i & 0 \\ -2 & -i \end{bmatrix} \begin{bmatrix} i & 0 \\ -2 & -i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Since $T^{*2}T^4 = (T^*T^2)^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, then $T \in 2$ power class Q.

$$T^3 = T^2T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} i & 2 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} -i & -2 \\ 0 & i \end{bmatrix}$$

$$T^6 = T^3T^3 = \begin{bmatrix} -i & -2 \\ 0 & i \end{bmatrix} \begin{bmatrix} -i & -2 \\ 0 & i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T^{*2}T^6 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T^*T^3 = \begin{bmatrix} -i & 0 \\ 2 & i \end{bmatrix} \begin{bmatrix} -i & -2 \\ 0 & i \end{bmatrix} = \begin{bmatrix} -1 & -2i \\ -2i & 3 \end{bmatrix}$$

$$(T^*T^3)^2 = \begin{bmatrix} -1 & -2i \\ -2i & 3 \end{bmatrix} \begin{bmatrix} -1 & -2i \\ -2i & 3 \end{bmatrix} = \begin{bmatrix} -3 & -4i \\ -4i & 5 \end{bmatrix}$$

Now, $T^{*2}T^6 \neq (T^*T^3)^2$ since $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} -3 & -4i \\ -4i & 5 \end{bmatrix}$.

Therefore $T \notin 3$ power class Q.

Example 4.4.6

Consider the operator $T = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ acting on a 2-dimensional complex Hilbert space.

$$T^* = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$T^{*2} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$T^3 = T^2T = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T^6 = T^3T^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T^{*2}T^6 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$T^*T^3 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$(T^*T^3)^2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

Now, $T^{*2}T^6 = (T^*T^3)^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, and therefore $T \in$ 3-power class Q.

Consequently,

$$T^4 = T^2T^2 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$T^{*2}T^4 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix}$$

$$T^*T^2 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 11 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$(T^*T^2)^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

$T^{*2}T^4 \neq (T^*T^2)^2$ since $\begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix} \neq \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$. Hence,$T \notin$ 2-power class Q operators.

Example 4.4.5 and Example 4.4.6 above show that 2-power class Q and 3-power class Q operators are independent.

Jibril, (2011) introduced (SN) operators, the class of operators whose squares are 2-normal. According to Jibril,(2011), this class (SN) is independent to the class of n-power normal where $n = 3$ introduced by Jibril (2008). Example 4.4.7 and Example 4.4.8 show that $3N$

and SN are independent.

Example 4.4.7

Let $T = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ be an operator acting on complex Hilbert space.

Then,

$$T^* = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

$$T^{*2} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T^4 = T^2T^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T^4T^{*2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T^{*2}T^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $T^4T^{*2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = T^{*2}T^4, T \in (SN)$

.

Now,

$$T^3 = T^2T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

$$T^3T^* = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

$$T^*T^3 = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

Since $T^3T^* = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \neq T^*T^3 = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, $T \notin 3N$.

Example 4.4.8 shows that there exist $3N$ operators which are not in SN.

Example 4.4.8

Let $T = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ be an operator acting on complex Hilbert space.

$$T^* = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$T^3 = T^2T = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T^3T^* = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$T^*T^3 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$T^3T^* = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = T^*T^3$$

Hence $T \in 3N$

Now $T^4 = T^2T^2 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$

$$T^{*2} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$T^4T^{*2} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$$

$$T^{*2}T^4 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} - & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix}$$

$$T^4T^{*2} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \neq T^{*2}T^4 = \begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix}.$$

Hence $T \notin SN$

Theorem 4.4.9 below states the relation between square normal operators and class $3N$ operators.

Theorem 4.4.9

The classes of square normal operators and $3N$ operators are independent.

Example 4.4.10 and Example 4.4.11 gives an example of a square normal operator which is not a $3N$ operator and $3N$ operator which is not a square normal respectively.

Example 4.4.10

Let T be given by $T = \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix}$ where $i \in \mathbb{C}$, then $T^* = \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix}$

$$T^2 = \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix} \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T^{*2} = \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T^2T^{*2} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T^{*2}T^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$T^2T^{*2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = T^{*2}T^2$ and hence square normal.

However,

$$T^3 = T^2T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix} = \begin{bmatrix} -i & 0 \\ -i & i \end{bmatrix}$$

$$T^3T^* = \begin{bmatrix} -i & 0 \\ -i & i \end{bmatrix} \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$$

$$T^*T^3 = \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} \neq \begin{bmatrix} -i & 0 \\ -i & i \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$$

Hence $T \notin 3N$

Example 4.4.11

Let $T = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ be an operator acting on complex Hilbert space.

$$T^* = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$T^3 = T^2T = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T^3T^* = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$T^*T^3 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$T^3T^* = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = T^*T^3$$

Hence $T \in 3N$

Now,

$$T^{*2} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$T^2T^{*2} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$T^{*2}T^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$T^2T^{*2} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \neq T^{*2}T^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Hence T is not a square normal operator.

Panayappan (2012) studied the relationship between n-normal operators and n-power class Q operators where every n-normal operator is an n-power class Q operator as seen in Theorem 4.4.12.

Theorem 4.4.12 : Panayappan, (2012)

If $T \in B(H)$ is n-normal, then $T \in$ n power class Q

Proof

Since T is n-normal, then from the definition,

$$T^*T^n = T^nT^* \tag{4.28}$$

multiplying equation (4.28) by T^* from left to obtain

$$T^*T^*T^n = T^*T^nT^* \tag{4.29}$$

Multiplying equation (4.29) by T^n from right to get

$$T^*T^*T^nT^n = T^*T^nT^*T^n \tag{4.30}$$

This yields,

$$T^{*2}T^{2n} = (T^*T^n)^2 \tag{4.31}$$

Hence $T \in$ n power class Q. $\square$

Example 4.4.13 shows that an operator of 2 power class Q need not be 2-normal.

Example 4.4.13

Consider operator $T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ acting on a three-dimensional complex space.

Then $T^* = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$T^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$T^{*2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T^4 = T^2T^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T^{*2}T^4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T^*T^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(T^*T^2)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly, $T^{*2}T^4 = (T^*T^2)^2$. Hence T is a 2 power class Q operator.

However,

$$T^*T^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and}$$

$$T^2T^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Therefore T is not 2 normal.

Theorem 4.4.14 states the relationship between class Q^* operators and 2-normal operators.

Theorem 4.4.14

Let T be a normal operator. If T is a 2-normal operator, then $T \in Q^*$.

Proof

Since T is a 2-normal operator, then,

$$T^*T^2 = T^2T^* \tag{4.32}$$

Multiply equation (4.32) by T^* from left to obtain,

$$T^*T^*T^2 = T^*T^2T^* \tag{4.33}$$

Equation (4.33) yields

$$T^{*2}T^2 = T^*TTT^* \tag{4.34}$$

Since T is normal, the right side of equation (4.34) becomes

$$T^{*2}T^2 = TT^*TT^* \tag{4.35}$$

Equation (4.35) becomes

$$\begin{aligned} T^{*2}T^2 &= TT^*TT^* \\ &= (TT^*)^2 \end{aligned} \tag{4.36}$$

Hence $T \in Q^*$.

Example 4.4.15 shows that if the normality of the 2-normal operator is ignored, then, T is not a class Q^* operator.

Example 4.4.15

Consider the operator $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Now, $T^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

T is 2-normal since,

$$T^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T^2T^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T^*T^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Here $T^2T^* = T^*T^2$.

On normality of T, observe that,

$$TT^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and,}$$

$$T^*T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore T is not normal.

Consequently, T is not a class Q^* operator since,

$$T^{*2} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T^{*2}T^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(TT^*)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Clearly, $T^{*2}T^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq (TT^*)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Theorem 4.4.16: Wanjala and Nyongesa, (2021)

Let $T \in B(H)$ be a normal operator. If $T \in Q^*$, Then T is a square normal operator.

Proof

Suppose $T \in Q^*$, then

$$\begin{aligned} T^{*2}T^2 &= (TT^*)^2 \\ &= TT^*TT^* \text{ (but T is normal)} \\ &= TTT^*T^* \\ &= T^2T^{*2} \end{aligned}$$

Now $T^{*2}T^2 = T^2T^{*2}$. Hence T is square normal.

Remarks 4.4.17

Square normal operators are not necessarily class Q^* operators. Example 4.4.18 shows a square normal operator which is not class Q^*.

Example 4.4.18

Let $T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ be an operator in $B(H)$.

$$T^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$T^{*2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T^2T^{*2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and}$$

$$T^{*2}T^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Since $T^2T^{*2} = T^{*2}T^2$, T is square normal.

Now,

$$TT^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(TT^*)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T^{*2}T^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Clearly,$(TT^*)^2 \neq T^{*2}T^2$. Hence $T \notin Q^*$.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATION

5.1 Introduction

This chapter presents the summary of the study findings in relation to the study objectives. It also suggests further research areas.

5.2 Conclusion

The main objective of this study was to characterize square normal and class Q^* operators in Hilbert spaces and also explore the relationship between these two classes of operators and other operators in Hilbert spaces. By the extension of the properties of the classical normal operators to square normal and class Q^* operators, these objectives were achieved.

In section 4.2, the study focused on characterizing the square normal operators. The study obtained the following findings in Proposition 4.2.1.1, Theorems 4.2.1.2 and 4.2.1.4 respectively.

(a) If $T \in B(H)$ is a normal operator, so are

  (i) T^{-1} , if it exists.

  (ii) T^*

  (iii) Any operator S which is unitarily equivalent to T

(b) If $T \in B(H)$ is a normal operator, for any $\lambda \in \mathbb{C}$, then,

  (i) $\lambda + T$ is square normal

  (ii) λT is square normal

This shows that square normal operators are closed under scalar addition and scalar multiplication. However, the square normal operators must be normal for this property to hold.

(c) If $T, S \in B(H)$ are square normal operators that commutes, then,

(i) $T + S$ is square normal

(ii) TS is square normal

As seen in Section 4.3 on properties of class Q^* operators, the study established that class Q^* operators are not convex. Additionally, the following are the main results as seen in Theorems 4.3.1.3, 4.3.1.7 and 4.3.1.11.

(a) If $T, S \in B(H)$ are two class Q^* operators that commute, then, their sum $T + S$ is also a class Q^* operator.

(b) The product TS of two class Q^* operators T, S is in the class Q^* only if every operators commutes with its adjoint.

(c) For normal operators in class Q^*, their direct sum and tensor product belong to the class Q^*.

In Section 4.4, the relationship between square normal operators, class Q^* and n-normal operators is studied. This study has established that;

(a) Every 2-normal operator (2N) is a square normal operator.

(b) 3-normal operators (3N) and square normal operators are independent.

(c) Every 2-normal operator is a class Q* operator, provided that it is normal.

(d) Class Q* operators are square normal, but the opposite is not necessarily true.

5.3 Suggestion for further research

This study suggests the following areas for further research.

(i) Spectral properties of square normal and class Q^* operators. These properties are eigen values and eigen vectors, resolvent set and spectral radius among others.

(ii) Application of Fugled-Putnam theorem on square normal and class Q^* operators.

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Appendix A

Post Graduate Study Research Clearance (CHUKA UNIVERSITY)

CHUKA UNIVERSITY

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REF: SD13/58010/22

14th February, 2024

Director
National Commission for Science Technology and Innovation
Off Waiyaki Way, Upper Kabete
P O Box 30623, 00100
Nairobi.

Dear Sir / Madam,

Edith Warue Nyaga

The above-named person is a *bona fide* student of Chuka University Pursuing PhD in Pure Mathematics proposal titled**: Characterization and Commutation Relations on Square Normal and Class Q* Operators in Hilbert Spaces**

Ms. Nyaga has defended at the Faculty level and is now expected to conduct research.
Any assistance accorded will be highly appreciated

Yours sincerely,

14 FEB 2024

Prof. Moses Muraya, Ph.D.

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Appendix B
Research authorization (CHUKA UNIVERSITY)

CHUKA UNIVERSITY

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CHUKA UNIVERSITY INSTITUTIONAL ETHICS REVIEW COMMITTEE

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Direct Line: 0772894438
Email: info@chuka.ac.ke,
Website: www.chuka.ac.ke

6th February, 2024

REF: CUIERC/ NACOSTI/471
TO: **Edith Warue Nyaga**

RE: Characterization and Commutation Relations on Square Normal and Class Q* Operators in Hilbert Spaces .

This is to inform you that *Chuka University IERC* has reviewed and approved your above research proposal. Your application approval number is *NACOSTI/NBC/AC-0812.* The approval period is 6th February, 2024 – 6th February, 2025.

This approval is subject to compliance with the following requirements;

i. Only approved documents including (informed consents, study instruments, MTA) will be used
ii. All changes including (amendments, deviations, and violations) are submitted for review and approval by *Chuka University IERC.*
iii. Death and life threatening problems and serious adverse events or unexpected adverse events whether related or unrelated to the study must be reported to *Chuka University IERC* within 72 hours of notification
iv. Any changes, anticipated or otherwise that may increase the risks or affected safety or welfare of study participants and others or affect the integrity of the research must be reported to *Chuka University IERC* within 72 hours
v. Clearance for export of biological specimens must be obtained from relevant institutions.
vi. Submission of a request for renewal of approval at least 60 days prior to expiry of the approval period. Attach a comprehensive progress report to support the renewal.
vii. Submission of an executive summary report within 90 days upon completion of the study to *Chuka University IERC.*

Prior to commencing your study, you will be expected to obtain a research license from National Commission for Science, Technology and Innovation (NACOSTI) https://oris.nacosti.go.ke and also obtain other clearances needed.

Yours sincerely

Dr. Benjamin Kanga
SECRETARY

Appendix C
Research Permit (NACOSTI)

REPUBLIC OF KENYA

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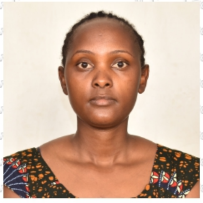

This is to Certify that Ms.. EDITH WARUE NYAGA of Chuka University, has been licensed to conduct research as per the provision of the Science, Technology and Innovation Act, 2013 (Rev.2014) in Tharaka-Nithi on the topic: Characterization and Commutation Relations on Square Normal and Class Q* Operators in Hilbert Spaces for the period ending : 14/March/2025.

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See overleaf for conditions

Appendix D
Publications

[1] Warue, E., Musundi, W., Kinyanjui J., (2024). *On some properties of Square Normal Operators*. Journal of Advances in Mathematics and Computer Science, 39(8):68-78.

[2] Warue, E., Musundi, W., Kinyanjui J., (2024). *On some properties of Class Q^* Operators*. Asian Journal of Pure and Applied Mathematics,6(1):201-211.