CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

EXAMINATION FOR THE AWARD OF DEGREE OF DEGREE OF MASTER OF SCIENCE IN MATHEMATICS (PURE)

MATH 804: GENERAL TOPOLOGY II

STREAMS: **TIME: 3 HOURS**

DAY/DATE: WEDNESDAY 13/12/2017 **8.30 A.M – 11.30 A.M**

INSTRUCTIONS:

- **Answer any three questions**
- **Do not write on the question paper**

QUESTION ONE: (20 MARKS)

(a) (i) When is a topological space X said to be connected? Explain why connectedness is a topological property. (2 mark)

(ii) Let Y be a subspace of X, define a separation of Y as a pair of disjoint non-empty sets A and B whose union is Y, neither of which contains a limit point of the other. Prove that the space Y is connected if there exists no separation of Y. (4 marks)

(iii) Prove that the union of a collection of connected subspaces of X that have a common point is connected. (4 marks)

(b) Define a path in a space X. Hence show that a path-connected space X is connected. (3 marks).

(c) Define a locally compact space X. Hence show that the real line $\mathbb{R}$ is a locally compact space whereas the set of rational numbers is not. (3 marks)

(d) Explain the basic idea behind the Stone-Cech Compactification. Under what condition does Stone-Cech Compactification coincide with Wallman Compactification? (4 marks)

QUESTION TWO: (20 MARKS)

(a) (i) Let X and Y be topological spaces. Define the product space on $X \times Y$. (1 mark)

(ii) Show that if $\mathcal{B}$ is a basis for the topology X and $\wp$ is a basis for the topology of Y, then the collection $\beta = \{B \times C: B \in \mathcal{B}\ and\ C \in \wp\}$ is a basis for the topology of $X \times Y$. (3 marks)

(b) Prove that a finite Cartesian product of connected spaces is connected (5 marks)

(c) Let X, Y be topological spaces. Define projection maps π_1 and π_2 such that $\pi_1: X \times Y \to X$ by $\pi_1(x, y) = x$ and $\pi_2: X \times Y \to Y$ by $\pi_2(x, y) = y$.
Show that the collection $S = \{{\pi_1}^{-1}(U): U\ is\ open\ in\ X\} \cup \{{\pi_1}^{-1}(V): V\ is\ open\ in\ Y\}$ is a subbasis for the product topology on $X \times Y$. (4 marks)

(d) Let X be a metric space with metric d. Define $\bar{d}: X \times X \to \mathbb{R}$ by the equation $\bar{d}(x, y) = \min\{d(x, y), 1\}$. Show that $\bar{d}$ is a metric that induces the same topology as d.

(6 marks)

(e) State without prove the UrysohnLemma (1 mark)

QUESTION THREE: (20 MARKS)

(a) Define the following spaces as used in topology

(i) Hausdorff space (1 mark)

(ii) Regular space (1 mark)

(iii) Normal space. (1 mark)

(i) A lindelof space (1 mark)

(ii) A completely regular space (1 mark)

(iii) A perfectly normal space (1 mark)

(b) State and prove the UrysohnMetrization Theorem. (14 marks)

QUESTION FOUR: (20 MARKS)

(a) When is a collection $\mathcal{B}$ of subsets of X said to be countably locally finite? Hence state without proof the Nagata-Smirnov metrization theorem (2mark)

(b) State and prove the Tychonoff Theorem. (10 marks)

(c) Define the Euclidean metric d on $\mathbb{R}^n$ by the equation

$$d(x,y) = \| x - y \| = [(x_1 - y_1)^2 + \cdots . + (x_n - y_n)^2]^{\frac{1}{2}}$$

and the square metric ρ by the equation $\rho(x,y) = Max\,\{|x_1 - y_1|, \ldots \ldots, |x_1 - y_1|\}$

Prove that the topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$. (8 marks)
