CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

EXAMINATION FOR THE AWARD OF DEGREE OF DEGREE OF BACHELOR OF EDUCATION (ARTS,SCIENCE) ,BACHELOR OF SCIENCE, BACHELOR OF ARTS (MATHS-ECONS)

MATH 312: REAL ANALYSIS I

STREAMS: **TIME: 2 HOURS**

DAY/DATE: THURSDAY 7/12/2017 **8.30 A.M – 10.30 A.M**

INSTRUCTIONS:

- **Answer Question ONE and ANY Other TWO Questions.**
- **Do not write on the question paper**

QUESTION ONE: (30 MARKS)

(a) Define the following terms as used in analysis
  (i) A neighborhood of a point $x_0 \in \mathbb{R}$ (1 mark)
  (ii) An interior point of a subset $A \subset \mathbb{R}$ (1 mark)
  (iii) A limit point x of a subset A of $\mathbb{R}$. (1 mark)

(b) Determine whether the set $A = \{x \in \mathbb{R}: 0 < x \leq 1\}$ is open or not in $\mathbb{R}$. (2 marks)

(c) Prove that if x and y are positive real numbers then
  (i) x + y is also positive (2 marks)
  (ii) x<y $\Rightarrow \frac{1}{y} < \frac{1}{x}$ (3 marks)

(d) Prove by analytic method that the sets $X \cap Y$ and $X \setminus Y$ are non-intersecting and that their union is the set X (5 marks)

(e) Prove that arbitrary intersection of closed sets is closed (3 marks)

(f) Show that if $\lim_{x \to x_0} f(x)$exists, then that limit is unique (3 marks)

(g) Use the P-test to determine the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)2^{2n-1}}$ (2marks)

(h) Define a function $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$by $d(x, y) = \exp\{|x - y|\}$. Determine whether d is a metric. (4 marks)

(i) Show that if $N_1$and N_2 are neighborhoods of p, then their intersections is also a neighborhood of p. (3 marks)

QUESTION TWO: (20 MARKS)

(a) (i) Consider three sets $X, Y \; and \; Z.$
Show that $Z \backslash (X \cap Y) = (Z \backslash X) \cup (Z \backslash Y)$ (3marks)

(ii) Hence prove that $X \backslash \cap_{A_\lambda} = \cup \, (X \backslash A_\lambda) for \;\; \lambda \, \epsilon \, \Lambda$ (4marks)

(b) Using the first principle, prove that

(i) $\lim_{x \to \infty} \frac{1}{x^3+1} = 0$ (3 marks)

(ii) $\lim_{n \to \infty} x^3 + x^2 + xsinx + 1 = \infty$ (3 marks)

(iii) $\lim_{n \to \infty} \left(\frac{3n+5}{7n+8}\right) = \frac{3}{7}$ (3marks)

(v) $\frac{\lim}{x - x_0} x^2 = x_0^2$ (4 marks)

QUESTION THREE: (20 MARKS)

(a) Show that there is no rational number whose square is *5.* (4 marks)

(b) Let S be a non-empty subset of R. Prove that the real number A is the sup A iff both the following conditions are satisfied

(i) $x \le A \;\; \forall \, x \subset S$

(ii) $\forall \, \varepsilon > 0 \;\; \exists \, x' \in S : A - \varepsilon < x' \le A$ (5 marks)

(c) Find the limit superior and limit inferior of the sequence
$X_n = cos\frac{n\pi}{2} + \frac{1}{n}\sin(\frac{2n+1}{2})\pi : n \in \mathbf{N}$ (6marks)

(d) Let $g: X \to Y and \; f: Y \to Z$g be two real valued functions such that
(i) g is continuous at $a \in X$
(ii) $g(a) = b \in Y$
(iii) f is continuous $b \in Z$.
Prove that then their composite function $(f \; 0 \; g)(x)$ is continuous. (5marks)

QUESTION FOUR: (20 MARKS)

(a) Prove that every convergent sequence is bounded. Using an appropriate counter example show that the converse of this is not necessarily true. (7marks)

(b) State without proof each of the following tests of convergence series
(i) D'Alembert's (Ratio) test (2marks)
(ii) Cauchy's Root test (2marks)

(iii) Cauchy's Integral test (2marks)

(c) (i)Show that a function which is uniformly continuous on a metric space is continuous on the same metric space (3 marks)

(ii) Determine whether the function $f(x) = x^2$ on the interval (0,1)is uniformly continuous or not (Hint: Take $x'_n = n$, $x''_n = n + \frac{1}{n}$) (4 marks)

QUESTION FIVE: (20 MARKS)

(a) (i) Define a countable set (1 mark)

(ii) Hence illustrate that the set of rational numbers between $[0, 1]$ is countable whereas the set of real numbers $\mathbb{R}$ is uncountable (7 marks)

(b) Consider the set $C[0,1]$ which represents real valued continuous functions on the interval [0,1].Define a function $d: C[0,1] \times C[0,1] \to R$ by $d(f, g) = \int_0^1 |f(x) - g(x)|dx$. Show that $(C[0,1], d)$ is a metric space. (6marks)

(c) Prove that Intersection of finitely many open sets in a metric space is open. (6 marks)
