CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

EXAMINATION FOR THE AWARD OF DEGREE OF BACHELOR OF SCIENCE (MATHEMATICS)

MATH 449: PROBABILITY THEORY
STREAMS: BSc **TIME: 2 HOURS**.

DAY/DATE:TUESDAY 10/12/2019 **8.30 PM-10.30 PM**

Instruction: Attempt Question 1 and any other two

Question 1 (30 marks)

a). i) Define a *field* (also known as *algebra*) as used in the measure and probability theory. State the conditions that a given filed say $\mathcal{A}$ must meet in order to be a field.

(4 marks)

ii) Consider a field denoted by $\mathcal{A}$ and let $A_1, A_2, \ldots, A_n \in \mathcal{A}$. Using the ideas in (i) above, show that $\bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} A_i \in \mathcal{A}$.

(5 marks)

iii) Suppose $\mathcal{A}$ is a class of sets containing Ω and satisfies

$$A, B \in \mathcal{A} \text{ implies } A \setminus B = AB^c \in \mathcal{A}.$$

Show that $\mathcal{A}$ is a field.

(5 marks)

b). Consider a probability measure defined as $(\Omega, \mathcal{F}, \mathbb{P})$. Required:

i) Explain each of the elements defined in the above space $(\Omega, \mathcal{F}, \mathbb{P})$.

(3 marks)

ii) Consider $A \in \Omega$, explain all the properties of $\mathbb{P}$. Use mathematical expressions.

(3 marks)

iii) If two dice are rolled once and we are interested in the events where the two numbers of that show up are equal (A_1), their sum are odd (A_2), , their sums are 13 (A_3). Apply the concept of a probability measure to come up with Ω, $\mathcal{F}$ and $\mathbb{P}$ respectively for this experiment.

(6 marks)

c). Suppose, $Y_1, Y_2, \ldots$ is a sequence of random variables with $E(Y_n) \longrightarrow \mu$ and $var(Y_n) \longrightarrow 0$. Show that $Y_n \longrightarrow \mu$ in probability.

(4 marks)

Question 2 (20 marks)

a). (Borel-Cantelli Lemma) Suppose, $A_1, A_2, \ldots$ is a sequence of events

i) If $\sum \mathbb{P}(A_n) < \infty$ then, $\mathbb{P}(A_n, \text{i.o}) = \mathbb{P}(\text{Lim Sup } A_n) = 0$

ii) If $\sum \mathbb{P}(A_n) = \infty$ and $A_1, A_2, \ldots$ are independent then $\mathbb{P}(A_n, \text{i.o}) = \mathbb{P}(\text{Lim Sup } A_n) = 1$.

Prove. (10 marks)

b). If

$$F(X) = P[X \leq x]$$

is continuous in x. Show that $Y = F(X)$ is measurable and that Y has a Uniform distribution

$$P[Y \leq y] = y, 0 \leq y \leq 1$$

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(10 marks)

Question 3 (20 marks)

a). Let X_n be iid, $E(X_n) = \mu$ and $\text{Var}(X_n) = \sigma^2$. Set $\bar{X} = \sum_{i=1}^{n} X_i/n$. Show that

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \xrightarrow{p} \sigma^2$$

(10 marks)

b). Suppose $X_k, x \geq 1$ are independent random variables and suppose that X_k has a gamma density $f_k(x)$

$$f_k(x) = \frac{x^{k-1}e^{-x}}{\Gamma(\gamma_k)}, x > 0, \gamma_k > 0$$

Give necessary and sufficient conditions for $\sum_{k=1}^{\infty} X_k$ to converge almost surely.

(10 marks)

Question 4 (20 marks)

Let E_n be events. Verify that

$$\sum_{k=1}^{n} 1_{E_k} = 1_{\cup_{k=1}^{n} E_k} \sum_{k=1}^{n} 1_k E_k$$

and hence use the Schwartz inequality to prove that

$$P(\cup_{k=1}^{n} E_k) \geq \frac{(E(\sum_{k=1}^{n} 1_{E_k}))^2}{E(\sum_{k=1}^{n} 1_{E_k})^2}.$$

(20 marks)

Question 5 (20 marks)

Suppose $T : (\Omega_1, B_1) \longmapsto (\Omega_2, B_2)$ is a measurable mapping and X is a random variable on Ω_1. Show that $X \in \sigma(T)$ iff there is a random variable Y on (Ω_2, B_2) such that

$$X(\omega_1) = Y(T(\omega_1)), \forall \omega_1 \in \Omega_1.$$

(20 marks)
