CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

FIRST YEAR EXAMINATION FOR THE AWARD OF DEGREE OF BACHELOR OF SCIENCE (MATHEMATICS)

MATH 406: FIELD THEORY

INSTRUCTIONS:

- Answer QUESTION 0NE AND ANY OTHER 2 questions
- Sketch maps and diagrams may be used whenever they help to illustrate your answer
- Do not write on the question paper
- This is a **closed book exam**, No reference materials are allowed in the examination room
- There will be **No** use of mobile phones or any other unauthorized materials
- Write your answers legibly and use your time wisely

QUESTION ONE (30 MARKS)

a) Let $\mathbb{Q}(\sqrt{2})=\{a + b\sqrt{2}\}$. Show that it is a subfield of $\mathbb{R}$. (5 marks)

b) Differentiate with examples between an algebraic and transcendental element over a field F (4 marks)

c) (i)Show that $\sqrt{2}+ \sqrt{3}$is algebraic over $\mathbb{Q}$

(ii) Find its degree of its extension over $\mathbb{Q}$ (6 marks)

d) Find a root of $x^4 + 4\ over\ \mathbb{Z}_5$ and factorize it fully in $\mathbb{Z}_5$ (5 marks)

e) Let E be a finite extension of degree n over a finite field F. Prove that if F has q elements the E has q^n elements (5 marks)

f) Construct a finite field GF(4) and the multliplication table of its nonzero elements (5 marks)

QUESTION TWO (20 MARKS)

a) Use the Einstein irreducibility criterion to show that $29x^5 + 42x^4 + 39x^3 - 12x^2 + 15x - 6$ is irreducible over Z[x] (5 marks)

b) By solving for the irreducible monic polynomial $f(x) \in Q(x)$ such that $\propto$ is a root of $f(x)$, find the degree of $\propto = \sqrt{\sqrt{5} - 2}$ over $\mathbb{Q}$. (4 marks)

c) Let E be an algebraic extension of field F and let $\alpha, \beta \in E$, explain what is meant by elements α and β being the conjugates over the field F and find all the conjugates of $\sqrt{1 + \sqrt{3}}$ over $\mathbb{Q}$. (6 marks)

d) Find the degree and basis for $\mathbb{Q}(\sqrt[3]{2}, \sqrt{5})$. (5 marks)

QUESTION THREE (20 MARKS)

a) Show that the field $F = Q(i, -i, \sqrt{5}, -\sqrt{5}$ is a simple extension given by $F' = Q(i + \sqrt{5})$ $(i.e\ F = F')$ (5 marks)

b) By considering an irreducible polynomial $f(x)$ over $\mathbb{Z}_2$ of degree 3 construct $GF(8)$. (5 marks)

c) Show that a field F is algebraically closed if every non-zero polynomial in $f(x)$ factors into linear factors. (5 marks)

d) Prove that a finite extension over a field F is an algebraic extension over F (5 marks)

QUESTION FOUR (20 MARKS)

a) Find the splitting field of $x^4 - 9x^2 + 14$. (5 marks)

b) Determine whether the polynomial $x^4 - 3x + 4$ is irreducible in $\mathbb{Q}$ by first checking if it has a rational root. Can we conclude its irreducible? (6 marks)

c) Show that $4x^3 + x^2 - x + 3$ is irreducible in $\mathbb{Q}[x]$ (5 marks)

d) By solving for the irreducible monic polynomial $f(x) \in \mathbb{Q}[x]$ such that $\propto$ is a root of $f(x)$, find the degree of $\propto = \sqrt{\sqrt[3]{2} + 3}$ over $\mathbb{Q}$. (5 marks)

QUESTION FIVE (20 MARKS)

a) Show that $x^2 - x - 1$ is solvable by radicals (4 marks)

b) (i)Explain the set of all automorphisms of $[\mathbb{Q}(\sqrt{2}, \sqrt{3})/ \mathbb{Q}]$

(ii)Draw and explain the corresponding subgroup and subfield diagrams (15 marks)