CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

FOURTH YEAR EXAMINATION FOR THE DEGREE OF BACHELORS OF EDUCATION SCIENCE/ARTS; BACHELORS OF SCIENCE GENERAL, BACHELORS OF ARTS GENERAL

MATH 304: COMPLEX ANALYSIS

STREAMS: ````as above```` Y3S2 **TIME: 2HRS**

INSTRUCTIONS:

- Answer question **ONE** and **TWO** other questions
- Any applicable theorem applied should be clearly stated

QUESTION ONE: (30 MARKS)

a) Determine the three roots of the complex number $z = -1 + i$ (3 marks)

b) Compute the $lim_{z \to 2+i} \frac{z^2 - 4z + 5}{z^3 - z - 10i}$ (3 marks)

c) Find all the points (if any) in the complex plane where the function $f(z) = x^3 + y + i(y^2 - x)$ is differentiable. Determine whether $f(z)$ is analytic at these points. (4 marks)

d) Solve the equation $e^{2z-1} = 1$ (4 marks)

e) Evaluate the following complex integrals

i. $\oint \frac{1}{(z^2+1)^2} dz$ inside the region bounded by the circle $|z|$=4 (4 marks)

ii. $\oint \frac{1}{z^2} dz$ inside the region bounded by the ellipse $(x-2)^2 + \frac{1}{4}((y-5)^2 = 1$ (3 marks)

f) Find the sum of the series $\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k}$ (Hint: it is a geometric series) (5 marks)

g) Expand $f(z) = \frac{1}{1-z}$in Taylor's series about $z = 2i$ (4 marks)

QUESTION TWO (20 MARKS)

a) Evaluate $\left(\frac{1+i}{\sqrt{2}}\right)^{1337}$ and write your answer in the form $x + iy$ (4 marks)

b) Verify that the function $u(x, y) = e^{-x} siny$ is harmonic and find its harmonic conjugate. Write the analytic function $f(z) = u + iv$ (6 marks)

c) Differentiate between poles, removable singularities and essential singularity using a n appropriate example in each case. (6 marks)

d) Express cot z and cosec z in terms of e^{iz} and e^{-iz} , hence show that $\cos ec^2 z - \cot^2 z = 1$

(4 marks)

QUESTION THREE(20 MARKS)

a) Express $z = \frac{\sqrt{1+x^2}+ix}{x-i\sqrt{1+x^2}}$ in the form $a+ib$, where a and b are real numbers hence Find the complex number z^3. (5 marks)

b) Evaluate the integral $\oint_\gamma f(z)dz$ where $f(z) = \frac{1}{(z-1)^2(2z-5)}$ and γ is the contour defined by

i. The rectangle defined by x=0,x=2,y=-1 and y=1

ii. The circle $|z-1| = 2$ (7 marks)

c) State without proof the residue theorem and use it to evaluate the integrals

i. $\oint_c e^{\frac{1}{z}}dz$ where c is $|z-2| = 3$

ii. $\oint_c \frac{5z^2-z+2}{z^2(z+1)(z+2i)}dz$ C;$|z-i| = 1.5$ (8 marks)

QUESTION FOUR (20 MARKS)

a) Find all solutions to the equation $sin\, z = 5$ (6 marks)

b) Find the set of points in which the function $f(z) = \frac{\bar{z}}{|z|^2}$ is analytic (6 marks)

c) From the polar form of the Cauchy-Riemann equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$, derive the corresponding Laplace equation in polar form (3 marks)

(ii)Show that the function $v(x,y) = \frac{x}{x^2+y^2}$ is harmonic and find an analytic function $f(z) = u(x,y) + iv(x,y)$ (5 marks)

QUESTION FIVE (20 MARKS)

a) Find the region of convergence of the series $\sum_{n=0}^{\infty}\frac{(z+4)^n}{(n+5)^4 8^{n+1}}$ (7 marks)

b) Expand $f(z) = \frac{1}{z^2-z}$ in Laurent series valid for $1 < |z-2| < 2$ (8 marks)

c) Evaluate the integral $\oint e^{\frac{3}{z}}dz$ inside the unit circle ; $|z| = 1$ (5 marks)