International Journal of Modern Mathematical Sciences, 2022, 20(1): 1-6

Modern Scientific Press

International Journal of Modern Mathematical Sciences **ISSN:2166-286X**

Journal homepage:www.ModernScientificPress.com/Journals/ijmms.aspx *Florida, USA*

Article

Norm- Attainability of Generalized Finite Operators on C*-Algebra

Amenya C. Sule, Musundi W. Sammy*, Kirimi Jacob

Department of Physical Sciences, Chuka University; P.O. Box 109-60400 Chuka, Kenya

* Author to whom correspondence should be addressed: swmusundi@chuka.ac.ke.com

Article history: Received 28 February 2022, revised 3 June 2022, Accepted 3 June 2022, Published 10 June 2022.

Abstract: Norm -attainability of elementary operators on Hilbert and Banach spaces have been Characterized by many mathematicians. However, there is little information on Norm-attainability of generalized finite operators on C*-algebra. A pair of bounded linear operators A, B on a complex Hilbert space H is called generalized finite operators if $||AX - XB - I|| \geq 1$ for each $x\epsilon B(H)$. This paper therefore determines the norm attainability of these generalized finite operators on C*-algebra when implemented by norm attainable operators A, B.

Keywords: Generalized finite operators; Norm attainability; C*-algebra; Complex Hilbert space

Mathematics Subject Classification (2010): 47C10; 47B38;

---

1. Introduction

Let H be a complex Hilbert space, $B(H)$ be the collection of bounded linear operators on H with inner product space and GF be the set of norm attainable generalized finite operators, the inner derivation is defined by $\delta_A\epsilon(X) = ||AX - XA||$, and the generalized derivation by $\delta_{AB}(X) = ||AX - XB||$, while the generalized finite operator $||AX - XB - I|| \geq 1$ is said to be norm attainable, if for every pair of operators $A, B\epsilon B(H)$ implementing the generalized finite operators are norm attainable and there exists a scalar q and some unit sequence Z_n such that $||Z_n||$=1, |q|=1 and $||(A-q)*Z_n|| < \frac{1}{n,}$ and $||(B-q)Z_n|| > \frac{1}{n}||$.

Definition 1.1 Involution on algebra, (Gelfand *et al*. 1943)

If A is an algebra, a mapping $*: A \rightarrow A$, defined by $x \rightarrow x*$ is called an involution on algebra A if it satisfies the following four conditions; $\forall\ x, y \epsilon A$.

i) $\overline{(x+y)^*} = x^* + y^*$
ii) $(\lambda x)^* = \lambda x^*$
iii) $(xy)^* = y^* x^*$
iv) $(x^*)* = x^{**} = x$

If A is a Banach algebra with an involution and, for every $\forall x \epsilon A$ $\|x^* x\| = \|x\|^2$, then A is called $C*{-}algebra$.

Example of C*-algebra (Gelfand *et al*. 1943)

Let $B(H)$ be a collection of bounded linear operators on a complex Hilbert space H, with inner product space, then $B(H)$ is a C*-algebra.

Definition 1.2 Generalized finite operators (Mecheri 2005)

Given pairs of operators $(A, B) \epsilon\ B(H) \times\ B(H)$: $||AX - XB - I|| \geq 1$ is a generalized finite operator

Definition 1.3 Norm attainable operator (Okelo 2020)

An operator $A \epsilon B(H)$ is said to be norm- attainable if for every unit vector $x \epsilon H$ it then follows $||Ax|| = ||A||$.

2. Main Results

Theorem 2.1 (Okelo 2018)

Let $S, T \epsilon B(H)$ if both S and T are norm attainable then the basic elementary operator M_{ST} is also norm attainable.

The lemma below gives the result on norm attainability of inner derivative.

Lemma 2.2

Let H be a complex Hilbert space, $B(H)$ be the collection of bounded linear operators on H with, inner product space and GF be the set of norm-attainable generalized finite operators, if there exists a scalar q and some sequence Z_n such that $||Z_n||$=1, |q|=1 and $||(A-q)*Z_n|| < \frac{1}{n,}$ $AX \rightarrow -AX$ then the inner derivation $\delta_A \epsilon GF$ is said to be norm-attainable.

Proof

We define inner derivative δ_A as $\delta_A(X) = ||AX - XA||$, from $||(A-q)*Z_n|| < \frac{1}{n,}$, when n≥1, then we will have,

$$\begin{aligned} ||AX - XA||^2 &= ||(A-q)^* X Z_n - Z_n||^2 - ||X(A-q)Z_n||^2 \\ &= ||(AX - qX)Z_n - Z_n||^2 - ||(AX - qX)\, Z_n\, ||^2 \\ &= ||\, (AX - qX)\, Z_n||^2 + 1 - \{||(AX - qX)Z_n||^2\} \\ &= ||(AX - qX)||^2\, ||Z_n||^2 + 1 - \{||(AX - qX||^2 ||Z_n||^2\} \\ &= ||AX||^2 - ||X||^2 |q|^2 + 1 - \{||AX||^2 - \left||X|\right|^2 |q|^2\, \} \\ &= ||AX - (-AX) + qX||^2 \\ &= ||AX + AX + qX||^2 \end{aligned}$$

For the positive square roots of the equation, the result is,

$$\begin{aligned} \left||AX - XA|\right| &= ||AX + AX + qX|| \\ &= ||2AX + qX|| \\ &= 2\left||AX|\right| + ||qX|| \\ &= 2\left||A|\right| + 1 \end{aligned}$$

Implying that $\left||AX - XA|\right| = 2\left||A|\right| + 1 = \delta_A$.

Since operator A is norm attainable, it then follows that the inner derivative δ_A is norm attainable. Next we give the conditions for norm attainability of generalized derivative δ_{AB}.

Lemma 2.3

Let H be a complex Hilbert space, $B(H)$ be the collection of bounded linear operators on H, with inner product space and GF be the set of norm-attainable generalized finite operators, the generalized derivative $\delta_{AB} \epsilon GF$ is norm attainable if there exists some scalar q and a unit sequence Z_n such that $||Z_n||$=1, $|q| = 1$, $||(A-q)$* Z_n||> $\frac{1}{n}$ and $||(B-q)\, Z_n$||> $\frac{1}{n}$

Proof

We define a generalized derivative δ_{AB} as $\delta_{AB}(X)$=$||AX - XB||$ for every $x \epsilon B(H)$

It then follows that

$$||AX - XB||^2 = ||(A-q)XZ_n - Z_n||^2 - \{||X(B-q)\,Z_n||^2\}$$
$$= \{||(AX-qX)Z_n||^2 - ||Z_n||^2\} - \{||(BX-qX)\,Z_n||^2\}$$
$$= ||(AX-qX||^2 - ||Z_n||^2 - \{||(BX-qX)||^2$$
$$= ||AX - qX - Z_n||^2 - ||(BX-qX)||^2$$
$$= ||AX - qX - Z_n - BX + qX||^2 \qquad \text{(i)}$$
$$= ||AX - BX - Z_n - qX + qX||^2$$
$$= ||AX - BX - Z_n||^2$$

For the positive square roots of the equation, the result is,

$$\big|\,|AX - XB|\,\big| = ||AX - XB - Z_n||$$
$$= \big|\,|AX|\,\big| - \big|\,|BX|\,\big| + 1$$
$$= \big|\,|A|\,\big| - \big|\,|B|\,\big| + 1$$

Implying that $||AX - XB|| = ||A|| - ||B|| + 1$ (ii)

From equation (i) we get the inequality

$$||AX - XB||^2 \geq ||AX - qX - Z_n - BX + qX||^2$$

Implying that,

$$||AX - XB|| \geq ||AX - BX - qX||$$
$$\geq \big|\,|A|\,\big| - \big|\,|B|\,\big| + 1 \qquad \text{(iii)}$$

For the reverse inequality, from equation (i), we have

$$||AX - XB||^2 \leq ||AX + qX + Z_n - BX - qX||^2$$
$$\leq ||AX - BX + Z_n + qX - qX||^2$$
$$\leq ||AX - BX + Z_n||^2$$

For the positive square root of the equation, the result is,

$$\big|\,|AX - XB|\,\big| \leq \big|\,|AX|\,\big| - \big|\,|BX|\,\big| + ||Z_n||$$
$$\leq \big|\,|A|\,\big| - \big|\,|B|\,\big| + 1 \qquad \text{(iv)}$$

From equation (iii) and (iv) we get

$$\big|\,|AX - XB|\,\big| = ||A|| - ||B|| + 1$$

Hence $\big|\,|AX - XB|\,\big| = ||A|| - ||B|| + 1 = \delta_{AB}$. Therefore δ_{AB} is norm attainable since A and B are norm attainable.

The next theorem gives the main results of our study on norm attainability of generalized finite operators.

Theorem 2.4

Let H be a complex Hilbert space, $B(H)$ be the collection of bounded linear operators on H with inner product space and $A, B \epsilon GF$, if A and B are norm attainable, then the generalized finite operators $(AB) \epsilon B(H) \times B(H)$: $||AX - XB - I|| \geq 1$ is norm attainable.

Proof

For the operators $A, B \epsilon B(H)$, it is known from lemma 2.3 that $||AX - XB|| = ||A|| - ||B|| + 1$

We let $||Z_n||$=1,$|q| = 1$, $||(A - q)^* Z_n|| > \frac{1}{n}$ and $||(B - q)\, Z_n|| > \frac{1}{n}$

Now for every n≥ 1, then we will have

$$\begin{aligned} ||AX - XB - I\,|| &\geq Sup\,\{||(AX - XB - I)Z_n||\} \\ &\geq Sup\,\{||(A - q)XZ_n - Z_n|| - ||X(B - q)\,Z_n|| + 1\} \\ &\geq Sup\,\{||A|| - ||B|| + 1\} \end{aligned}$$

Implying that $||AX - XB - I|| \geq ||A|| - ||B|| + 1$ (i)

For the reverse inequality,

$$\begin{aligned} ||AX - XB - I\,|| &\leq Sup\,\{||(A - q)XZ_n - Z_n|| - ||X(B - q)\,Z_n|| + 1\} \\ &\leq Sup\,\{||AX|| + |q|||X|| - [||BX|| + |q|||X||] + 1\} \\ &\leq Sup\,\{||A|| - ||B|| + 1 - 1 + 1 \\ &\leq Sup\,\{||A|| - ||B|| + 1\} \end{aligned}$$

Implying that, $||AX - XB - I|| \leq ||A|| - ||B|| + 1$ (ii)

From equation (i) and (ii) we get

$$\big|\,|AX - XB - I|\big| = ||A|| - ||B|| + 1$$

Therefore the generalized finite operator $A, B \epsilon B(H)$: $||AX - XB - I|| \geq 1$ is norm attainable.

3. Conclusion

The generalized finite operators $(AB) \epsilon B(H) \times B(H)$: $||AX - XB - I|| \geq 1$ is norm attainable when implemented by norm attainable operators A, B.

References

[1] Blanco, A., Boumazgour, M., & Ransford, T. J.; On the Norm of Elementary Operators. *Journal of the London Mathematical Society*, *70*(2)(2004): 479-498.

[2] Fujimoto, I. A Gelfand–Naimark Theorem for C*-algebras. *Pacific Journal of Mathematics*, *184*(1)(1998): 95-119.

[3] Kawira, E., Kingangi, D. & Musundi, S. W. *On The Norm of an Elementary Operator of Finite Length in AC* Algebra*, (2018).

[4] Kinyanjui, J. N., Okelo, N. B., Ongati, O., & Musundi, S. W. Norm Estimates for Norm-Attainable Elementary Operators. *International Journal of Mathematical Analysis*, *12*(3) (2018): 137-144.

[5] Mecheri, S. Generalized Finite Operators. *Demonstration Mathematica*, *38*(1)(2005): 163-168.

[6] Okelo B. Norm-Attainability and Range-Kernel Orthogonality of Elementary Operators. *Communications in Advanced Mathematical Sciences*, *1*(2) (2018): 91-98.

[7] Okelo, N. B., Agure, J. O., & Ambogo, D. O. Norms of Elementary Operators and Characterization of Norm-attainable Operators. *Int. J. Math. Anal*, *24*(2010): 1197-1204.

[8] Okelo, B. N. *The Norm Attainability of Some Elementary Operators*, (2012).

[9] Okelo, N. B. On Norm-Attainable Operators in Banach Spaces. *Journal of Function Spaces*, (2020).

[10] Okwany, I. O., Okelo, N. B., & Ongati, O. Norm Estimates for Hermitian Derivations. *Pure and Applicable Analysis*, *2022*(2022): 2-2.

[11] Stampfli, J. The Norm of a Derivation. *Pacific Journal of Mathematics*, *33*(3)(1970): 737-747.

[12] Timoney, R. M. Some formulae for norms of elementary operators. *Journal of Operator Theory*, (2007): 121-145.