CHUKA **UNIVERSITY**

UNIVERSITY EXAMINATIONS

FOURTH YEAR EXAMINATION FOR THE AWARD OF

BACHELOR OF SCIENCE DEGREE IN MATHEMATICS

MATH 405: ALGEBRA II

STREAMS: "AS ABOVE"

TIME: 2 HOURS

DAY/DATE: WEDNESDAY 31/3/2021 **11.30 AM – 1.30 PM**

INSTRUCTIONS:

- Answer Question **ONE** and any other **TWO** Questions
- Sketch maps and diagrams may be used whenever they help to illustrate your answer
- Do not write anything on the question paper
- This is a **closed book exam**, No reference materials are allowed in the examination room
- There will be **No** use of mobile phones or any other unauthorized materials
- Write your answers legibly and use your time wisely

QUESTION ONE (30 MARKS)

a) Verify whether or not the following are ideals in the given ring
   i. R is the ring of rational numbers and I is the set off non negative rational numbers
   ii. R is Z[x] and I is the set of polynomials in Z[x] whose leading coefficient is even
   iii. R is Z_6 and I is the set of elements in Z_6 of the form $r + Z_6$ where r is an even number

(6 marks)

b) The addition and part of the multiplication table for the ring R={a,b,c} are given below. Use the distributive laws to complete the multiplication table below

+	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

*	a	b	c
a	a	a	a
b	a	c	
C	a		

(5 marks)

c) Working in Q[x],find the highest common factor of $x^3 + x^2 - 8x - 12$ and $x^3 + 5x^2 + 8x + 4$ and express it as a linear combination of the two functions (5 marks)

d)) If R is a commutative ring with identity, show that R[x] is also a commutative ring with identity (5 marks)

e) Let R be the ring of all 2X2 matrices over Z with the usual addition and multiplication of matrices.

   i. Show that the subset of R consisting of all matrices of the form $T = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| a, b, c \in Z \right\}$ is a non-commutative subring with unity.

   ii. Which elements of T ate invertible?

   iii. Find if $I = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a, b \in Z \right\}$ is an ideal of T (6 marks)

QUESTION TWO (20 MARKS)

a) Consider the set $R = \{[0],[2],[4],[6],[8],[10].[12],[14],[16]\} \subseteq Z_{18}$.

   i. Construct addition and multiplication tables for R using operations as defined in Z_{18} (2 marks)

   ii. Show that R is a commutative ring with unity. (2 mars)

   iii. Show that R a subring of Z_{18} (2 marks)

   iv. Does R have zero divisors? (1 marks)

   v. Is R a field? If yes illustrate each element with its inverse (1 mark)

b) Let P be an ideal in R. P is a prime ideal if and only if $\frac{R}{P}$ is an integral domain. (6 marks)

c) Let M be an ideal in R. M is a maximal ideal iff $\frac{R}{M}$ is a _field. (6 marks)

<u>QUESTION THREE (20 MARKS)</u>

a) Let F be a field, and let f(x) and g(x) be polynomials in F[x] where F is a field

   i. Prove that deg(fg) =deg(f) +deg(g). (4 marks)

Consider the polynomials $f(x) = 2x^2 + 3x + 3$ and g(x) = 3x + 1inthe polynomial ring $Z_6[x]$s. Find:

   i. deg(f)

   ii. deg(g)

   iii. deg(fg)

   iv. why is the theorem above not satisfied (4marks)

b) Let X be a non-empty set and R be the setoff all subsets of X. Define addition and multiplication in R as follows

$A + B = A \cup B - A \cap B$

$A * B = A \cap B$

For all $A \in R$ define a function $f : R \to Z_2$ as $f(x) = \begin{cases} \bar{1} if x \in A \\ \bar{0} otherwise \end{cases}$

i. Show that $A + \phi = A$ and $A + A = \phi$ (5 marks)

ii. Show that f is a homomorphism of rings (7marks)

QUESTION FOUR (20 MARKS)

a) Let U be a fixed non-empty set and R be the set of subsets of U with addition and multiplication defined by $A + B = A \cup B$ and $A \times B = A \cap B$. Verify whether or not $(R,+,\times)$ is a ring. (6 marks)

b) Let F be a field, and f(x) anon-zero polynomial in F[x]. Prove the following

i. If g(x) ∈ F[x] is an associate of f(x),then deg(g) =deg(f). (4 marks)

ii. There exists a unique monic polynomial that is an associate of f(x). (4 marks)

c) Let I and J be ideals in the ring Z of integers, Verify whether or not

i. $I \cup J$ is an ideal

ii. $I \cap J$ is an ideal (6 marks)

QUESTION FIVE (20 MARKS)

a) i. Use the Euclidean Algorithm to find $hcf(x^3 + 2x^2 - x - 2, x^2 - 4x + 3)$ in Q[x]. (4 marks)

ii. Hence, or otherwise, find polynomials s,t in Q[x] for which

x – 1= $s(x^3 + 2x^2 - x - 2) + t(x^2 - 4x + 3)$ (4 marks)

iii. find $lcm(x^3 + 2x^2 - x - 2, x^2 - x - 2)$ (4 marks)

b) prove that in a ring of integers, every ideal is a principal ideal (8 marks)

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