CHUKA



UNIVERSITY EXAMINATIONS

FOURTH YEAR EXAMINATION FOR THE DEGREE OF BACHELORS OF EDUCATION SCIENCE/ARTS AND BACHELOR OF SCIENCE MATHEMATICS

MATH 403: MEASURE THEORY

STREAMS: B. ED (SCIENCE/ARTS), BSC. MATH

DAY/DATE: TUESDAY 21/09/2021 INSTRUCTIONS:

- Answer question **ONE** and **TWO** other questions
- This is a **closed book exam**, No reference materials are allowed in the examination room
- There will be **No** use of mobile phones or any other unauthorized materials

QUESTION ONE: (30 MARKS)

- a) Prove that if $\mu^*(A) = 0$ then for any set B, $\mu^*(A \cup B) = \mu^*(B)$. (3 marks)
- b) Prove the following properties of an outer measure μ^*
 - i. $\mu^*(\phi) = 0$ (2 marks)
 - ii. $\mu^*(\{x\}) = 0$ (2 marks)
 - iii. If $A \subseteq B$ then $\mu^*(A) \le \mu^*(B)$ (2 marks)
- c) Using the properties of outer measure, prove that
 - i. The unit interval I = [0,1] is not countable
 - ii. The outer measure of all the irrational numbers in I = [0,1] is 1. (4 marks)
- d) Define a measurable function and show that the characteristic function on a measurable set is measurable.
 (3marks)
- e) Differentiate a finite measure and sigma finite measure. (2marks)
- f) Show that the space (R, B, μ) is not complete, where μ is the restriction of Lebesgue measure to the Borel sets. (4 marks)

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TIME: 2 HOURS

8.30 A.M. – 10.30 A.M.

g) Shoe that if $A \cup B$ is measurable whenever A and B are measurable, then $A \cap B$ is measurable (4marks)

h) Show that the integral is monotone i.e.

i).If
$$f, g \in M^+(X, x)$$
 and $f \leq g$ then $\int f d\mu \leq \int g d\mu$ (2 marks)

ii) If
$$f \in M^+(X, x)$$
 and $E, F \in x$ such that $E \subset F$ then $\int_E f d\mu \leq \int_F f d\mu$ (2marks)

QUESTION TWO: (20 MARKS)

- a) Define a Lebesgue measurable subset of R.
- b) Define a Lebesgue non-measurable. Hence show that if a set F is Lebesgue non-measurable, there exist a proper subset A of F such that $0 < \mu^*(A) < \infty$ (4 marks)
- c) Show that if $\mu^*(A) = 0$, then A is measurable hence or otherwise show that a countable set is measurable. (5 marks)

d) Prove that measurable sets form a sigma algebra (9 marks)

QUESTION THREE: (20 MARKS)

- a) Let X, Y be non-void sets and f: X → Y be a function. Let ⊃ be the σ- algebra of subsets of Y and let x = {f⁻¹(E): E ∈ ⊃}. Prove that then x is the σ- algebra of subsets of X (6marks)
- b) Let A be an uncountable subset of R and define a class Ω of subsets of A as follows: $\Omega = \{E \subseteq A \text{ if } E \text{ is countable or } A \in E \text{ is countable}\}$
 - i. Show that Ω is a sigma algebra

ii. Define a function $f: \Omega \to R$ as $f(E) = \begin{cases} 0 if E = countable \\ 1 otherwise \end{cases}$.

Show that f is a measure

QUESTION FOUR: (20 MARKS)

- a) Let f be a measurable function, prove that the following conditions are equivalent
 - i. $\{x: f(x) > \alpha\}$ is Lebesgue measurable $\forall \alpha \in R$
 - ii. $\{x: f(x) \ge \alpha\}$ is Lebesgue measurable $\forall \alpha \in R$
 - iii. $\{x: f(x) < \alpha\}$ is Lebesgue measurable $\forall \alpha \in R$
 - iv. $\{x : f(x) \le \alpha\}$ is Lebesgue measurable $\forall \alpha \in R$ (8 marks)
- b) Show that if f is measurable, then so are the functions f^2 and |f|. Is the converse true? Verify (4 marks)

(8 marks)

. . .

(2 marks)

(6 marks)

c) (i)State without prove the monotone convergence theorem (M.C.T) (2 marks) (ii)Show that the sequence $f_n(x) = \frac{1}{n} \chi_{[0,n]}$ for $n \in N$ uniformly converges to f = 0(2 marks)

(iii)Show that M.C.T does not apply in the sequence $f_n(x) = \frac{1}{n} \chi_{[0,n]}$ for $n \in N$. Explain your answer. (4 marks)

QUESTION FIVE: (20 MARKS)

a) (i)Show that intervals of the form (a,b): a < b and $a, b \in R$ are Lebesgue measurable

(8 marks)

(ii) Hence conclude that the sets [a,b], [a,b), (a,b] are Lebesgue measurable (6 marks)

b) Let $\{E_n\}$ be a sequence of measurable sets with the properties $E_n \supseteq E_{n+1}$ and $\mu(E_1) < \infty$.

Prove that
$$\mu(\bigcap_{n=1}^{\infty} E_n) = \lim_{n \to \infty} E_n$$
 (6 marks)

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