CHUKA UNIVERSITY

UNIVERSITY EXAMINATION

RESIT /SPECIAL EXAMINATION

EXAMINATION FOR THE AWARD OF DEGREE OF BACHELOR OF SCIENCE

MATH 401: MEASURE THEORY

STREAMS: **TIME: 2 HOURS**

DAY/DATE: WEDNESDAY 03/11/2021 **2.30 P.M – 4.30 P.M**

INSTRUCTIONS:

ANSWER ALL THE QUESTIONS

QUESTION ONE: (30 MARKS)

a) Prove that a sigma algebra is closed under countable intersections. (5 marks)

b) Define a counting measure as follows: for any countable set E, $\mu(E) = \textit{number of elements of } E$. note that $\mu(E) =$ n if E has n elements and $\mu(E) = \infty$ if E has infinitely many elements.

c) Show that μ is a measure. (5 marks)

d) Prove the following properties of an outer measure μ^*

i. $\mu^*(\phi) = 0$ (3 marks)

ii. $\mu^*(\{x\}) = 0$ (3 marks)

iii. If $A \subseteq B$ then $\mu^*(A) \leq \mu^*(B)$ (3 marks)

e) Prove that if $\mu^*(A) = 0$, then A is measurable (5 marks)

f) Define a measurable function and show that the characteristic function on a measurable set is measurable. (5marks)

g) Define a Lebesgue integrable function (2 marks)

QUESTION TWO: (20 MARKS)

a) Let X, Y be non-void sets and $f: X \to Y$ be a function. Let $\beth$ be the σ- algebra of subsets of Y and let $\mathfrak{x} = \{f^{-1}(E): E \in \beth\}$. Prove that then $\mathfrak{x}$ is the σ- algebra of subsets of X (6marks)

b) Let A be an uncountable subset of R and define a class Ω of subsets of A as follows:
$\Omega = \{E \subseteq A$ if E is countable or A-E is countable$\}$

i. Show that Ω is a sigma algebra (6 marks)

ii. Define a function $f: \Omega \to R$ as $f(E) = \begin{cases} 0 if E = countable \\ 1 otherwise \end{cases}$.

Show that f is a measure (8 marks)

QUESTION THREE: (20 MARKS)

a) Let $(X, \mathfrak{x})$ be a measurable space and $f: X \to \mathbb{R}^*$ be a given function. Show that the following statements are equivalent

i. $\{x \in X: f(x) > a\}(= f^{-1}(a, \infty]) \in \mathfrak{x}$ for all $a \in \mathbb{R}^*$

ii. $\{x \in X: f(x) \geq a\}(= f^{-1}[a, \infty]) \in \mathfrak{x}$ for all $a \in \mathbb{R}^*$

iii. $\{x \in X: f(x) < a\}(= f^{-1}[-\infty, a)) \in \mathfrak{x}$ for all $a \in \mathbb{R}^*$

iv. $\{x \in X: f(x) \leq a\}(= f^{-1}[-\infty, a]) \in \mathfrak{x}$ for all $a \in \mathbb{R}^*$ (8 marks)

b) Explain a uniform convergence sequence of functions (2 marks)

(ii)Show that the sequence $f_n(x) = \frac{1}{n}\chi_{[0,n]}$ for $n \in N$ uniformly converges to $f = 0$

(3 marks)

(iii)Show that M.C.T does not apply in the sequence $f_n(x) = \frac{1}{n}\chi_{[0,n]}$ for $n \in N$. Explain your answer. (4 marks)

(iv) State Fatous' Lemma and show that it applies in this case (3 marks)
