

CHUKA



UNIVERSITY

UNIVERSITY EXAMINATIONS

**SECOND YEAR EXAMINATION FOR THE DEGREE OF BACHELORS OF
SCIENCE MATHEMATICS AND BACHELOR OF EDUCATION (SCIENCE)**

MATH 405: ALGEBRA II

STREAMS: AS ABOVE

TIME: 2 HOURS

DAY/DATE: MONDAY 27/09/2021

11.30 A.M. – 1.30 P.M.

INSTRUCTIONS:

- Answer Question **ONE** and any other **TWO** Questions
- Sketch maps and diagrams may be used whenever they help to illustrate your answer
- Do not write anything on the question paper
- This is a **closed book exam**, No reference materials are allowed in the examination room
- There will be **No** use of mobile phones or any other unauthorized materials
- Write your answers legibly and use your time wisely

QUESTION ONE (30 MARKS)a) Find two ideals I and J in the ring Z of integers such that

- i. $I \cup J$ is an ideal
- ii. $I \cup J$ is NOT an ideal (4 marks)

b) The addition and part of the multiplication table for the ring $R = \{a, b, c\}$ are given below. Use the distributive laws to complete the multiplication table below

+	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

*	a	b	c
a	a	a	a
b	a	c	
C	a		

(4 marks)

b) For any element $a \in \mathbb{Z}$; the ring of integers let $[a]_6$ denote $[a] \in \mathbb{Z}_6$ and $[a]_2$ denote $[a] \in \mathbb{Z}_2$

i. Prove that the mapping $\phi : \mathbb{Z}_6 \rightarrow \mathbb{Z}_2$ defined by $\phi([a]_6) = [a]_2$ is a homomorphism

ii. Find $\ker \phi$ (6 marks)

c) Working in $\mathbb{Q}[x]$, find the highest common factor of $x^3 + x^2 - 8x - 12$ and $x^3 + 5x^2 + 8x + 4$ and express it as a linear combination of the two functions (5 marks)

d) If R is a commutative ring with identity, show that $R[x]$ is also a commutative ring with identity

(5 marks)

e)

i. Show that the set of R consisting of all matrices of the form

$$T = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\} \text{ is a non-commutative ring with unity.}$$

ii. Which elements of T are invertible?

iii. Find if $I = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$ is an ideal of T (6 marks)

QUESTION TWO (20 MARKS)

a) Consider the set $R = \{[0], [2], [4], [6], [8], [10], [12], [14], [16]\} \subseteq \mathbb{Z}_{18}$.

i. Construct addition and multiplication tables for R using operations as defined in \mathbb{Z}_{18} (2 marks)

ii. Show that R is a commutative ring with unity. (2 marks)

iii. Show that R is a subring of \mathbb{Z}_{18} (2 marks)

- iv. Does R have zero divisors? (1 mark)
- v. Is R a field? If yes illustrate each element with its inverse (1 mark)
- b) Let R be a ring in which the only ideals are $\{0\}$ and R . Prove that R is a field (6 marks)
- c) Let I be an ideal of a commutative ring R with unity. Show that if I contains a unit element, then $I=R$ (6 marks)

QUESTION THREE (20 MARKS)

- a) Let F be a field, and let $f(x)$ and $g(x)$ be polynomials in $F[x]$ where F is a field
 - i. Prove that $\deg(fg) = \deg(f) + \deg(g)$. (4 marks)

Consider the polynomials $f(x) = 2x^2 + 3x + 3$ and $g(x) = 3x + 1$ in the polynomial ring $Z_6[x]$

s. Find:

- i. $\deg(f)$
- ii. $\deg(g)$
- iii. $\deg(fg)$
- iv. why is the theorem above not satisfied (4 marks)
- b) Let X be a non-empty set and R be the set of all subsets of X . Define addition and multiplication in R as follows

$$A + B = A \cup B - A \cap B$$

$$A * B = A \cap B$$

For all $A \in R$ define a function $f : R \rightarrow Z_2$ as $f(x) = \begin{cases} \bar{1} & \text{if } x \in A \\ \bar{0} & \text{otherwise} \end{cases}$

- i. Show that $A + \phi = A$ and $A + A = \phi$ (5 marks)
- ii. Show that f is a homomorphism of rings (7 marks)

QUESTION FOUR (20 MARKS)

- a) Let U be a fixed non-empty set and R be the set of subsets of U with addition and multiplication defined by $A + B = A \cup B$ and $A \times B = A \cap B$. Verify whether or not $(R, +, \times)$ is a ring. (6 marks)
- b) Let F be a field, and $f(x)$ a non-zero polynomial in $F[x]$. Prove the following
 - i. If $g(x) \in F[x]$ is an associate of $f(x)$, then $\deg(g) = \deg(f)$. (4 marks)

- ii. There exists a unique monic polynomial that is an associate of $f(x)$. (4 marks)
- c) Let I and J be ideals in the ring Z of integers, Verify whether or not
 - i. $I + J = \{x + y : x \in I, y \in J\}$ is an ideal
 - ii. $I \cap J$ is an ideal (6 marks)

QUESTION FIVE (20 MARKS)

- a) i. Use the Euclidean Algorithm to find $hcf(x^3 + 2x^2 - x - 2, x^2 - 4x + 3)$ in $Q[x]$. (4 marks)
- ii. Hence, or otherwise, find polynomials s, t in $Q[x]$ for which

$$x - 1 = s(x^3 + 2x^2 - x - 2) + t(x^2 - 4x + 3)$$
 (4 marks)
- iii. find $lcm(x^3 + 2x^2 - x - 2, x^2 - 4x + 3)$ (4 marks)

- b) prove that in a ring of integers, every ideal is a principal ideal (8 marks)

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