

CHUKA



UNIVERSITY

UNIVERSITY EXAMINATIONS

**FOURTH YEAR EXAMINATION FOR THE AWARD OF
DEGREE OF BACHELOR OF EDUCATION SCIENCE / ARTS, BSC. MATHEMATICS**

MATH 409: FUNCTIONAL ANALYSIS

STREAMS: AB5 / AB1 ; EB6 Y4S2

TIME: 2 HOURS

DAY/DATE : WEDNESDAY 22 /09/ 2021

2.30 PM – 4.30 PM

INSTRUCTIONS TO CANDIDATES:

- Answer question **ONE** and **TWO** other questions
- Do not write on the question paper
- This is a **closed book exam**, No reference materials are allowed in the examination room
- There will be **No** use of mobile phones or any other unauthorized materials
- Write your answers legibly and use your time wisely

QUESTION ONE: (30 MARKS)

- (a) Distinguish the following terms
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| i. A convergence sequence and a Cauchy sequence | (2marks) |
| ii. Holders Inequality and Minikowsk`'s Inequality | (2marks) |
| iii. A Hamel base and Schauder Basis | (2marks) |
| iv. A semi-norm and a para-norm | (4marks) |
| v. An Iteration and a contraction mapping | (2marks) |
| vi. A complete space and a compact space | (2 marks) |
- (b) (i) Define a Banach space. Hence give any two examples of Banach spaces (2 marks)
- (ii) When are two norms said to be equivalent on a vector space? (2marks)
- (c) (i) When are two normed linear spaces said to be Isometrically Isomorphic? (1mark)
- (ii) Hence show that the spaces $C_0^* \sim \ell_1$ are Isometrically Isomorphic (3 marks)

- (d) Prove that for a weakly convergent sequence, its limit point of is unique (3 marks)
- (e) (i) Define a sesquilinear functional on normed linear spaces X and Y (2 marks)
- (ii) Hence state without proof the Riesz's Representation Theorem (3 marks)

QUESTION TWO: (20 MARKS)

- (a) State the Parallelogram law as used in inner product spaces. Hence using an appropriate example, illustrate that all Banach spaces are not necessarily inner product spaces. (6 marks)
- (b) Define a fixed point of a mapping T of a set X . Give two cases that illustrate a fixed point mapping. (3marks)
- (c) State and prove the Banach Fixed Point Theorem on a metric space X (11marks)

QUESTION THREE: (20 MARKS)

- (a) Prove that on the space of all sequences S , the mapping defined by
- (i) $P_1(x) = \sup |x_n| \forall n \geq 1$ is a semi-norm (3marks)
- (ii) $p(x) = |x| = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_k|}{1+|x_k|}$ is a paranorm (5marks)
- (b) Define a norm on a linear space X . Hence show that the mapping $\| \cdot \| : R^n \rightarrow R$ defined by
- $$\| x \|_{\infty} = \left(\left(\sum_{k=1}^n \| x_k \|^2 \right) \right)^{\frac{1}{2}}$$
- is a normed space. (12marks)

QUESTION FOUR: (20 MARKS)

- (a) Let $T: X \rightarrow Y$ be a linear operator from a normed linear space X into a normed linear space Y , prove that T is continuous if and only if T is bounded (10 marks)
- (b) Prove that strong convergence implies weak convergence, and with an appropriate counter example show that the converse is not necessarily true (10 marks)

QUESTION FIVE: (20 MARKS)

(a) Let $T: X \rightarrow Y$ be a linear operator from a normed linear space X into a normed linear space Y . Prove that :

(i) T is continuous iff T is bounded. (5 marks)

(ii) T is continuous at the origin implies that T is uniformly continuous on X (3 marks)

(b) Show that if $(T_n)_1^\infty$ be a sequence of bounded linear operators each defined on a Banach space X into a normed linear space Y such that for each $x \in X$, $\lim_{n \rightarrow \infty} T_n(x) = T(x)$ exists in Y , then T is a bounded linear operator from X into Y . (8marks)

(c)) Define a meager subset of a metric space X . Hence state without proof, Baire's Category Theorem (4marks)

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