CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

SECOND YEAR EXAMINATION FOR THE AWARD OF DEGREE OF SCIENCE IN MATHEMATICS, BACHELORS OF ARTS (MATHS-ECONOMIS)

MATH 206: INTRODUCTION TO REAL ANALYSIS

STREAMS: AS ABOVE (Y2S2) **TIME: 2 HOURS**

DAY/DATE: WEDNESDAY 07/07/2021 **5.00 P.M. – 7.00 P.M.**

INSTRUCTIONS:

- Answer question **ONE** and **TWO** other questions
- Sketch maps and diagrams may be used whenever they help to illustrate your answer
- Do not write on the question paper
- Write your answers legibly and use your time wisely

QUESTION ONE: (30 MARKS)

(a) Find if the following sets are bounded or not and if bounded find the $sups$ and $infs$

(i) $S_1 = \{x \in \mathbb{R}: 3 \leq x < 7\}$ (3 marks)

(ii)$S_2 = \left\{1 + (-1)^n \frac{1}{n}: n \in \mathbf{N}: \right\}$ (3 marks)

(iii) $S_2 = \{1 + (-1)^n . n: n \in \mathbf{N}: \}$ (3 marks)

(b) Determine the accumulation points of each of the set of real numbers

(i) The set of natural numbers **N**;

(ii) $(a, b]$

(iii) The set of irrational points (3 marks)

(c) Let $A \subseteq \mathbb{R}$ be given by $A = \{x \in \mathbb{R}: 0 < x \leq 1\}$. Show that the element $\frac{1}{2} \in A$ is an interior point of A whereas 1 is not (4 marks)

(d) Show that the subset$A \subset \mathbb{R}$ is closed if and only if $A = \bar{A}$ (4 marks)

(e) Show that the set of squares of whole numbers is countable (3 marks)

(f) Let S be a non-empty subset of R. Prove that the real number A is the supremum of S if and only if both the following conditions are satisfied

(i) $x \leq A \quad \forall\, x \subset S$

(ii) $\forall\, \varepsilon > 0 \quad \exists\, x' \in S : A - \varepsilon < x' \leq A$ (4 marks)

(g) State without proof the following properties for real numbers.

(i) Completeness axiom (2 marks)

(ii) Archimedean Property (1 mark)

QUESTION TWO: (20 MARKS)

(a) Prove that $\sqrt{\mathrm{n}+1} - \sqrt{\mathrm{n}-1}$ for any integer $n \geq 1$ is an irrational number(4 marks)

(b) Given that $x, y, z \in \mathbb{R}$. Show that,

(i) If $x \neq 0$ and $xy = xz$ then, $y = z$ (4 marks)

(ii) If $x < y$ then, $\frac{1}{y} < \frac{1}{x}$ (4 marks)

(iii) $x < y$ if and only if $x^2 < y^2$ (4 marks)

(c) Prove that the set of real numbers $\mathbb{R}$ is uncountable (4 marks)

QUESTION THREE: (20 MARKS)

(a) Using the $\varepsilon - \delta$definition of limit of a function, prove that

(i) $\lim_{n\to\infty}\left(\frac{(-1)^n}{n+5}\right) = 0$
(4 marks)

(ii)$\lim_{\mathrm{x}\to 2}(\mathrm{x}^3 + \mathrm{x} - 10) = 0$ (4 marks)

(iii)$\lim_{x\to\infty} e^{2x} = \infty$ (4 marks)

(iv) $\lim_{n\to\infty}\left(\frac{3n-7}{7n+9}\right) = \frac{3}{7}$ (4 marks)

(b) Using the first principle show that the derivative of the function $y = x^3 - 5x$ is $3x^2 - 5$ (4 marks)

QUESTION FOUR: (20 MARKS)

(a) Prove that if the limit of a function exists then that limit is unique (4 marks)

(b) Determine whether the function $f(x) = x^2$ is contious at the point x=1;

$f(x) = \begin{cases} x & 0 \leq x < 1 \\ \frac{1}{2}x & 1 \leq x < 2 \end{cases}$ is continuous at x=1 (4 marks)

(c) Using the function $f(x) = x^{\frac{1}{3}}$ at the point$x = 0$, show that the property of continuity does not necessarily imply differentiability. (7 marks)

(d) Define an open subset A of $\mathbb{R}$. Hence show that A is open iff $A = A^0$ (5 marks)

QUESTION FIVE: (20 MARKS)

(a) Define a Cauchy sequence (x_n) in R. Hence prove that if a sequence (x_n) is convergent then it is Cauchy (6 marks)

(b) Let (x_n) be a sequence of real numbers prove that if $x_n \to x$ then, $|x_n| \to |x|$. (4 marks)

(c) When is a sequence (x_n) said to be bounded?. Hence prove that every convergent sequence is bounded, but the converse of this does not always hold. (10 marks)

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