CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

SECOND YEAR EXAMINATION FOR THE DEGREE OF BACHELORS OF SCIENCE MATHEMATICS

MATH 305: ALGEBRA I

STREAMS: `` As above``

TIME: 2HRS

DAY/DATE: ……………………. ………………………..

INSTRUCTIONS:

- Answer Question **ONE** and any other **TWO** Questions
- Sketch maps and diagrams may be used whenever they help to illustrate your answer
- Do not write anything on the question paper
- This is a **closed book exam**, No reference materials are allowed in the examination room
- There will be **No** use of mobile phones or any other unauthorized materials
- Write your answers legibly and use your time wisely

QUESTION ONE (30 MARKS)

a) Let *be a binary operation on the set of positive integers Determine whether or not * is Commutative and Find an identity element with respect to * if it exists

a) $a * b = c$, where c is the smallest integer between a and b

b) $a * b = c$, where c is 5 more than $a + b$ (4 marks)

a) Let $SL(2, Z)$ denote the set of all 2x2 matrices with integer coefficients, whose determinant is one. Verify whether or not $SL(2, Z)$ is a group under matrix multiplication . (5 marks)

c) Given a group G, define the centre of G,(z(G)) and show that it is a normal subgroup of G (4 marks)

d) Let G and H be groups and $\emptyset: G \to H$ be a homomorphism. Define the kernel of Ø. Hence show that a homomorphism $\emptyset: G \to H$ is injective if and only if $ker\emptyset = \{e\}$ (4 marks)

(2 marks)

e) Suppose a,b and c are elements of an integral domain D such that ab=ac and $a \neq 0$. Prove that b=c (3 marks)

f) Verify whether or not the following statements are true about groups

i. A group of order 21 has a subgroup of order 5 (2 marks)

ii. A group of order 7 is abelian (3 marks)

iii. Show that the set S of permutation mappings given below forms a cyclic group.

g) $I = \begin{pmatrix} 1 & 2 & 34 \\ 1 & 2 & 34 \end{pmatrix}, A = \begin{pmatrix} 1 & 2 & 34 \\ 2 & 3 & 41 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 34 \\ 3 & 4 & 12 \end{pmatrix}, C = \begin{pmatrix} 1 & 2 & 34 \\ 4 & 1 & 23 \end{pmatrix}$(5 marks)

QUESTION TWO (20 MARKS)

a) (i) Express the product $(1\ \ 2\ \ 7\ \ 3\ \ 4)(5\ \ 6)$ in S_7as a single permutation in matrix form

(2marks)

(ii) Express $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 5 & 9 & 4 & 2 & 1 & 6 & 3 \end{pmatrix}$ as a product of disjoint cycles in S_9

(2marks)

(iii) Given that $h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ is a permutation in S_4. Show that $h^{-1} \circ h = e$

(3marks)

b) Prove that every permutation can be written as as product of transpositions

(5 marks)

c) Let n be a positive integer. Define $\phi : (Z,+) \to (Z_n,+_n)$ as $\phi(k) = \overline{k} \quad ,k \in Z$ and where $\overline{k}$ denotes the remainder of division of k by n.

i. Show that ϕ is a group homomorphism (3 marks)

ii. Find ker ϕ (2 marks)

iii. Find the index $[Z : \ker\phi]$ (2 marks)

QUESTION THREE (20 MARKS)

a) Let G be a group in which every element has order at most 2. Show that G is abelian (3 marks)

b) Show that in an abelian group G, the set of all elements with finite order in G is a subgroup of G. (5 marks)

c) Let G be the set of eight elements given by $G = \{\pm 1, \pm i, \pm j, \pm k\}$ with multiplication given by $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

   i. Construct a multiplication table for the group. (6 marks)

   ii. Consider the cyclic group group $H = < 1, -1 >$. list all the distinct cosets of H in G. Is H a normal subgroup of G? (6marks)

QUESTION FOUR (20 MARKS)

a) Let G be a group with identity e. Prove that:

   (i) If $a, b \in G$, then $(ab)^{-1} = b^{-1}a^{-1}$ (2 marks)

   (ii) If $(ab)^2 = a^2b^2 \ \ \forall\, a, b \in G$, then G is abelian (3 marks)

b))Let H be a subgroup of G. Show that the following statements are equivalent.

   (i) H is a normal subgroup of G

   (ii) $xHx^{-1} = H \ \ \forall\, x \in G$

   (iii) $xH = Hx \ \ \forall\, x \in G$

   (iv) $(xH)(yH) = xyH$ (10 marks)

c) Let $G = \langle \mathbb{R}^+, \times \rangle$, the group of positive real numbers under multiplication and $H = \langle \mathbb{R}, + \rangle$, the additive group of real numbers. Show that the mapping given by $\emptyset(x) = \ln x$ is an Isomorphism. (5 marks)

QUESTION FIVE (20 MARKS)

a) State without proof Sylow's theorems (6 marks)

b) Using the theorems in (a) above, show that a group of order 15 is cyclic (8 marks)

c) Prove that a group of order p^2;where p is prime is abelian (6 marks)