CHUKA **UNIVERSITY**

UNIVERSITY EXAMINATIONS

FIRST YEAR BLOCK 1II EXAMINATION FOR THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS (PURE)

MATH 813: BANACH ALGEBRA I

STREAMS: SB/PT **TIME: 3 HOURS**

DAY/DATE: WEDNESDAY 7/4/2021 **2.30 PM – 5.30PM**

INSTRUCTIONS:

- Answer ANY THREE Questions.
- Do not write on the question paper.

QUESTION ONE: (20 MARKS)

(a) Show that if Ω is a locally compact Haursdoff's space denoted by $M_w=\{w \in \Omega : f(w)=0\}$, then is a modular ideal (4 marks)

(b) (i) Let A be a Banach algebra and $x \in A$ with $\| e-x \| < 1$. Show that the inverse of x

exists and $\| x^{-1} \| \leq \frac{1}{e-\| e-x \|}$

(6marks)

(ii) Prove that for a Banach algebra , the set of invertible elements $Inv(A)$ is an open set

(3marks)

(c) Prove that every positive linear functional f on a Banach algebra A with an involution has the following properties

(i) $f(x^{¿})=\overline{f(x)}$

(ii) $¿ f(x y^{¿}) \vee ¿^2 \leq \overline{f(x x^{¿}) f(y y^{¿})} ¿$

(iii) $¿ f(x) \vee ¿^2 \leq f(e) . f(x x^{¿}) ¿$ (7 marks)

QUESTION TWO: (20 MARKS)

(a) If Ω is a topological space and $C_b(\Omega)$ the space of bounded continuous complex valued functions on Ω, with $\| f \| = ^{|f(x)|, \forall \in \Omega}$ ¿. Show that $C_b(\Omega)$ is a normed algebra.

(5 marks)

(b) Prove that if I is a closed ideal in a normed algebra A, then $\frac{A}{I}$ is a normed algebra when endowed with the quotient norm $\| a+I \| = \{ inf \| a+a^{-1} \| : a^{-1} \in I \}$ (5 marks)

(c) Define a multiplicative linear functional on the algebra A. Hence prove that the norm of a multiplicative linear functional is 1 (6 marks)

(d) If A is a non-unital Banach algebra and $\mu : A \rightarrow C$ is a multiplicative linear functional on A, show that $|\mu(x)| \leq \| x \|, \forall x \in A$ (4 marks)

QUESTION THREE: (20 MARKS)

(a) Suppose $x \in A$ ($C^{¿}$ algebra). Prove that:

(i) If x is self adjoint, then $Sp(x) \subset R$ (3 marks)

(ii) If x is unitary, then $Sp(x) \subset \delta D = \{ x \in C, \| x \| \leq 1 \}$ (3 marks)

(iii) An involution mapping $x \rightarrow x^{¿}$ defined on $A \rightarrow A$ is an isometry, i.e. preserves the norm (2 marks)

(b) Prove that the Maximal Ideal space of $L^{'}(z)$ is homomorphic to $\delta D, \delta D = \{ Z \in C \}, |Z| = 1$ (12 marks)

QUESTION FOUR: (20 MARKS)

(a) Prove that for a Banach algebra A and $\forall x \in A$, the spectrum $Sp(x)$ is a non-empty set.

(7 marks)

(b) Let a be an element of a unital algebra A. Prove that the spectral radius of a,

$r(a) = \lim_{n \rightarrow \infty} \| a^n \|^{\frac{1}{n}}$ (6 marks)

(c) Let A be a Banach algebra with an involution and $\forall x \in A$, prove that

(i) $x+x^{¿}, i\left(x+x^{¿}\right) \wedge xx^{¿}$ are self adjoint

(ii) $x=u+iv$ has a unique representation for u and v being self adjoint

(iii) If x is invertible in A, then $\left(x^{¿}\right)^{-1}=\left(x^{-1}\right)^{¿}$ (7 marks)

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