CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

SECOND YEAR EXAMINATION FOR THE AWARD OF MASTER OF SCIENCE (PURE MATHEMATICS)

MATH 812: FIELD THEORY

STREAMS: MSC (MATH) **TIME: 3 HOURS**

DAY/DATE: TUESDAY 06/04/2021 **2.30 P.M – 5.30 P.M.**

INSTRUCTIONS:

- Answer ANY three questions
- Sketch maps and diagrams may be used whenever they help to illustrate your answer
- Do not write on the question paper
- This is a **closed book exam**, No reference materials are allowed in the examination room
- There will be **No** use of mobile phones or any other unauthorized materials
- Write your answers legibly and use your time wisely

QUESTION ONE (20 MARKS)

a) i. Define a ring homomorphism (2 marks)

ii. Show that $\phi : C \to M_2(R)$ given by $\phi(a+bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ for all $a, b \in R$is a ring homorphism. (3 marks)

b) Let P and Qbe primary ideals in a commutative ring R, then show that the product PQ of P and Qdefined by $PQ = \sum_{i=1}^{n} p_i q_i \vee p_i \in P, q_i \in Q$ is an ideal of commutative ring R. (4 marks)

c) Define the ring of fractions of ring R and suppose $R = Z_6 \wedge S = \{1,5\}$, find the ring of fractions $S^{-1}R$. (5 marks)

d) i. Define a primary ideal. (2 marks)

ii. Show that if $Q_1, Q_2, \ldots, Q_n$ are primary ideals in a commutative ring R, all of which are primary for the prime ideal P, then $\cap_{i=1}^{n} Q_i$ is also a primary ideal belonging to P.

(4 marks)

QUESTION TWO (20 MARKS)

a) i. Define the terms maximal and prime ideal. (2 marks)

ii. Find all the prime ideals and maximal ideals of Z_{12}. (4 marks)

b) Prove that if R is a commutative ring with unity, then I is a maximal ideal of R if and only if $\frac{R}{I}$ is a field. (6 marks)

c) Prove that if F is a field, then $F[x]$ is a principal ideal domain. (4 marks)

d) Show that the field $F=Q¿$ is a simple extension given by $F'=Q¿$

(4 marks)

QUESTION THREE (20 MARKS)

a) State the Einstein irreducibility criterion and hence show that $17x^3+5x^2+15x-5$ is irreducible over Q (4 marks)

b) By solving for the irreducible monic polynomial $f(x) \in Q(x)$ such that $\propto$ is a root of $f(x)$, find the degree of $\propto = \sqrt{\sqrt{7}+3}$ over Q. (3 marks)

c) Find all the conjugates of $\sqrt{1+\sqrt{2}}$ over Q. (5 marks)

d) By considering an irreducible polynomial $f(x)$ over Z_2 of degree 3 construct $GF(8)$.

(5 marks)

e) Define a simple extension K of a subfield F *if* E is a field extension of F and $\alpha \in E$ and hence give an example of a simple field extension of the field or real numbers. (3 marks)

QUESTION FOUR (20 MARKS)

a) Find the splitting field of x^4-5x^2+6. (4 marks)

b) Prove that if the element r is algebraic over the field F with minimum polynomial $f(x) \in F[x]$ then $o¿$. (6 marks)

c) Show that a field F is algebraically closed if and only if every non-zero polynomial in $f(x)$ factors into linear factors. (6 marks)

d) Show that if I is an ideal and if I has a primary decomposition then I has a reduced or normal primary decomposition. (4 marks)
